Recent Results for the Panex Puzzle

By Nick Baxter for G4G5, April 2002, revised November 2008

Introduction

In 1983, the TRICKS Company in Japan introduced the Panex Puzzle. It is a sliding piece puzzle, similar in appearance to the well-known Tower of Hanoi, but with a twist. It has achieved legendary status within some circles because, unlike the Tower of Hanoi, the best algorithm has not yet been proven to be optimal.

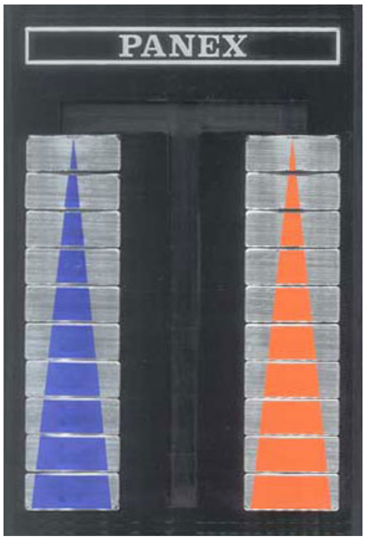

Fig. 1. **Panex Silver**

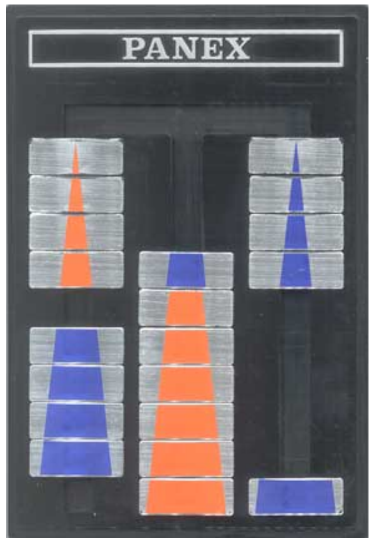

Fig 2. **Example position**

Fig. 3. **Panex Gold, with packing retainer**

The tray contains an E-shaped network of channels, with two stacks of pieces on opposite sides. The primary goal is to swap the positions of the two sets of pieces. (The secondary, warm-up challenge is to just move one stack from the side to center channel.) The catch is that the channels include a hidden mechanism that restricts the movement of the pieces–no piece can ever move lower than it's starting position (see figure 2).

There are two versions of the puzzle: Gold and Silver. The Silver includes very friendly graphics (see figure 1) that visually indicate the starting level of each piece. The Gold version (see figure 3) is not so friendly.

In late 1983, Mark Manasse, Danny Sleator, and Victor Wei wrote an unpublished paper [5] where they present a "good algorithm" for solving Panex, and use this to determine upper bounds for solving the level-5 through level-10 variations. They also describe a computer program that searches for the optimal solution. This program confirmed the algorithm as optimal for level-5 and level-6, but ran out of computing power for higher levels.

For the level-10 Panex, they prove a theoretical lower bound of 27,564 moves; but is generally believed that the upper bound of 31,537 given by their algorithm is the best possible result.

Recent Results

In addition to [5], Panex is discussed in a number of other puzzle publications (Vladimir Dubrovsky [2], Edward Hordern [4], and Slocum & Botermans [6]), and is implemented as a Java applet on the Internet (separately by David Bagley [1] and Emmett Henderson [3]).

Earlier this year, David Bagley wrote a program that follows the algorithm and program described in [5]. On Feb 7, 2002, he confirmed the optimal solution for level-7; and on March 26, 2002, he confirmed the optimal solution for level-8. In November 2004, he confirmed the optimal solution for level-9. The computing time currently estimated as needed to confirm the algorithm for level-10 is now down to two years; and in two years that estimate will likely be reduced to a time that will finally be practical!

Foundations

The algorithm for solving Panex described by Manasse, et al in [5] is built on three specific subtasks:

- *Tower* (T_n) - moves a level-n tower from a side channel to the center channel.
- *Sink* (S_n) - swaps a level-n (or higher) tile from a side channel with a level n-1 tower in the center channel.
- *X* - exchanges the position of the top two tiles in the center channel.

T_n and S_n are defined recursively for n>2, and are illustrated in figures 4 and 5 (shown for n=6). Also, some obvious variations of these operations will be used: ${}_LT_n$ and ${}_LS_n$ denote the operations using the left channel, and ${}_RT_n$ and ${}_RS_n$ use the right channel. Inverse notation, T^{-1} and S^{-1}, is used for the reverse sequence of moves.

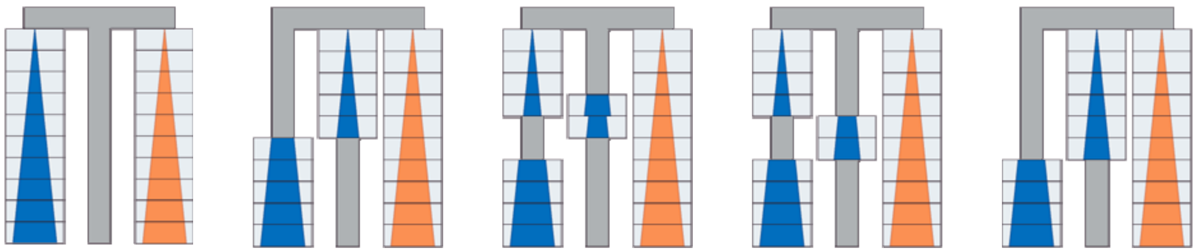

Fig 4-1. Before T_n | Fig. 4-2. After T_{n-1} | Fig. 4-3. After S_{n-1} | Fig. 4-4. After X | Fig. 4-5. After T_{n-2}

Figure 4. $\mathbf{T_n = T_{n-1} + S_{n-1} + X + T_{n-2}}$

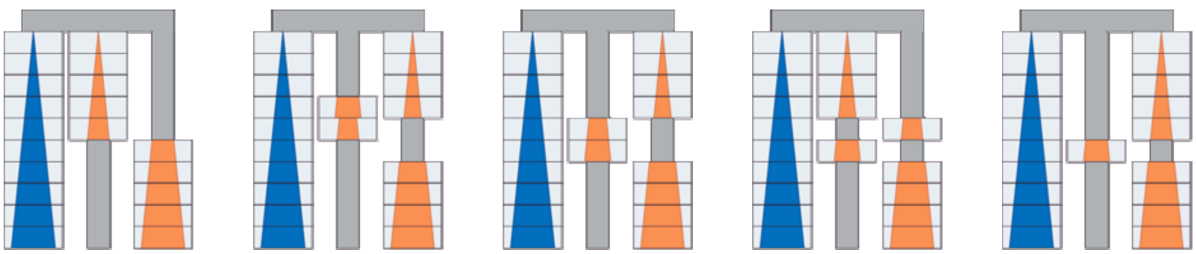

Fig 5-1. Before S_n | Fig 5-2. After S_{n-1} | Fig 5-3. After X | Fig. 5-4. After S^{-1}_{n-1} | Fig. 5-5. After T^{-1}_{n-2}

Figure 5. $\mathbf{S_n = S_{n-1} + X + S^{-1}_{n-1} + T^{-1}_{n-2}}$

We also define one additional maneuver not explicitly described in [5], which will greatly simplify the description of the algorithm:

- *Zigzag* (Z_n) - moves a level-n tile from one outside channel to the other, where the top of the center and opposite channels both contain level n-1 towers.

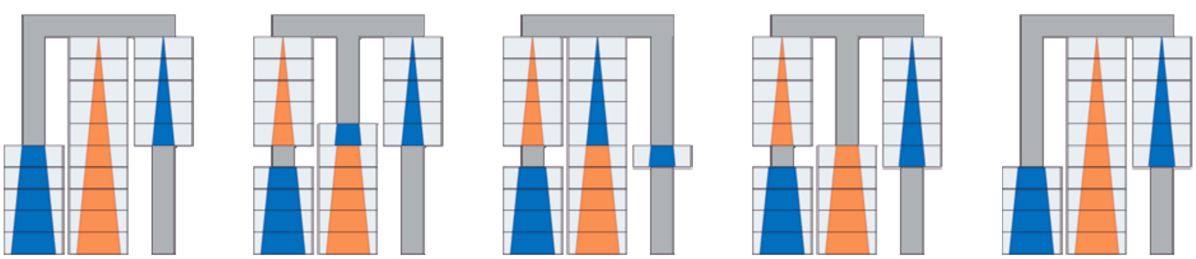

Fig 6-1. Before Z_n | Fig 6-2. After ${}_LS_n$ | Fig 6-3. After ${}_RS^{-1}_n$ | Fig. 6-4. After ${}_RT^{-1}_{n-1}$ | Fig. 6-5. After ${}_LT_{n-1}$

Figure 6. $\mathbf{Z_n = {}_LS_n + {}_RS^{-1}_n + {}_RT^{-1}_{n-1} + {}_LT_{n-1}}$

A "Good" Algorithm

The algorithm for solving Panex described modestly in [5] uses slight variations of T_n and S_n (where the last move is omitted in anticipation that the next maneuver will just move it back), and other subtle tricks and optimizations to squeeze out the best possible move count. For the sake of clarity, this description will ignore these refinements, and present the essence of the algorithm so that is it as clear as possible. We will see that even this version of the algorithm is still quite "good".

The algorithm starts out by moving both level-10 tiles to the center channel, then moving the blue level-10 tile to its goal position in the right channel.

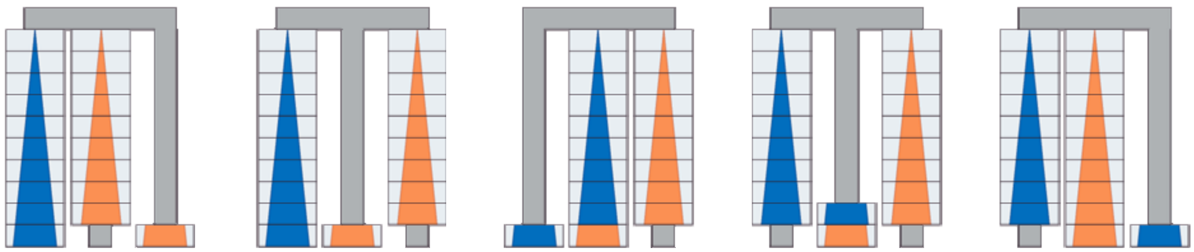

Fig 7-1. After ${}_RT_9$ Fig 7-2. After ${}_RS_{10}$ Fig. 7-3. After ${}_LT_9$ Fig. 7-4. After ${}_LS_{10}$ Fig. 7-5. After ${}_RS^{-1}{}_{10}$

Figure 7. **Getting Started**

The logical next step is to use ${}_RT^{-1}{}_9$ and ${}_LS^{-1}{}_{10}$ to get both level-10 tiles in their goal positions (see figure 8). Then after ${}_LT^{-1}{}_8$, the puzzle is reduced to a level-9 problem which can be solved recursively (this position will be comparable to that shown in figure 7-2 but one level shorter). Unfortunately, this leads to an inferior solution, using about 30% more moves than necessary.

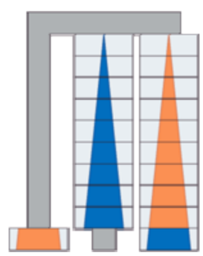

Fig 8. **A recursive approach**

The clever approach discovered by Manasse, et al is to use Z_n (for n=1 to 9) to individually, move the blue tiles from the left channel to the right. The benefit of this is that we don't use another expensive level-10 (S_{10} or T_{10}) maneuver. Starting from the position in figure 7-5:

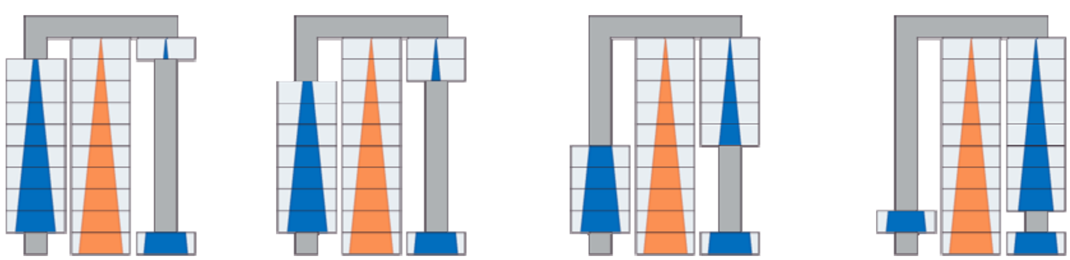

Fig 9-1. After Z_1 Fig 9-2. After Z_2 Fig. 9-3. After $Z_3 + Z_4 + Z_5$ Fig. 9-4. After $Z_6 + Z_7 + Z_8$

Figure 9. **Using Z_n to move blue tiles**

Z_9 will move the final blue tile to the right channel. But in this special case we do not use the very last portion of Z_9 ($_LT_8$) because it happens to be the reverse of what would have been the next set of moves (see Figure 10-1). Now the end is in sight; with an just obvious reorganization of the orange tiles, the puzzle is solved:

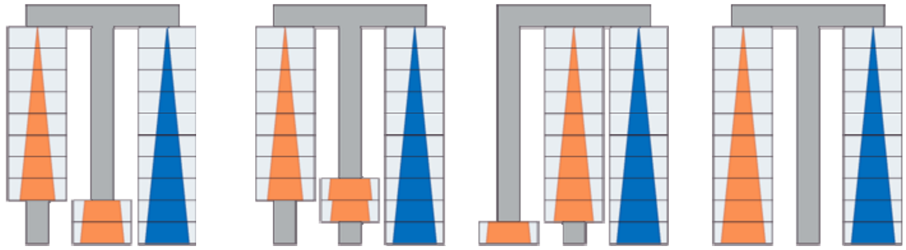

Fig. 10-1. After partial Z_9 Fig 10-2. After X Fig 10-3. After $_LS^{-1}{}_9$ Fig. 10-4. After $_LT^{-1}{}_9$

Figure 10. **Final Steps**

Summary

Figure 11-1 shows the move counts using the optimal algorithm presented in [5] (X_n is the proven best solution, U_n is that predicted by the algorithm). Figure 11-2 shows the move counts using the simplified algorithm presented here (U_n), and as improved simply by removing consecutive canceling moves as described at the top of page 3 (U'_n).

n	T_n	S_n	X_n	U_n
1	1	1	3	
2	3	3	13	
3	9	8	42	
4	24	23	128	
5	58	58	343	343
6	143	142	881	881
7	345	345	2,189	2,189
8	836	835	5,359	5,359
9	2,018	2,018	13,023	13,023
10	4,875	4,874		31,537

Figure 11-1. Move counts (X_n) from [5], extended by David Bagley

n	T_n	S_n	U_n	U'_n
1	1	1		
2	3	3		
3	9	8	47	43
4	24	23	135	131
5	60	59	361	348
6	147	146	917	886
7	357	356	2,271	2,195
8	864	863	5,551	5,365
9	2,088	2,087	13,481	13,030
10	5,043	5,042	32,637	31,544

Figure 11-2. Move counts for simplified algorithm

Manasse, et al [5] write about some interesting topics that are beyond the scope of this paper: they prove that the algorithms for T_n and S_n are in fact optimal, give recurrence relationships and closed formulas for T_n and U_n, and give an instructive description of the search program designed to take advantage of the unique symmetries of the Panex puzzle.

An edited and annotated version of [5] on my web site. The URL for my Panex page is http://www.baxterweb.com/puzzles/panex/.

References

[1] David Bagley, *Panex Applet*, http://gwyn.tux.org/~bagleyd/java/PanexApp.html.

[2] Vladimir Dubrovsky, "Nesting Puzzle, Part 1: Moving Oriental Towers", *Quantum*, Jan/Feb 1996, pp 53-57, 49-51.

[3] Emmett Henderson, *Panex Puzzle* (level 4 only), http://www.cheesygames.com/panex/ (and uncredited at numerous other sites).

[4] L.E. Hordern, *Sliding Piece Puzzles*, Oxford University Press 1986, pp 144-145, 220.

[5] Mark Manasse, Danny Sleator, and Victor K. Wei, *Some Results on the Panex Puzzle*, unpublished.

[6] Jerry Slocum & Jack Botermans, *Puzzles Old & New*, Plenary Publications International, 1986, p 135.