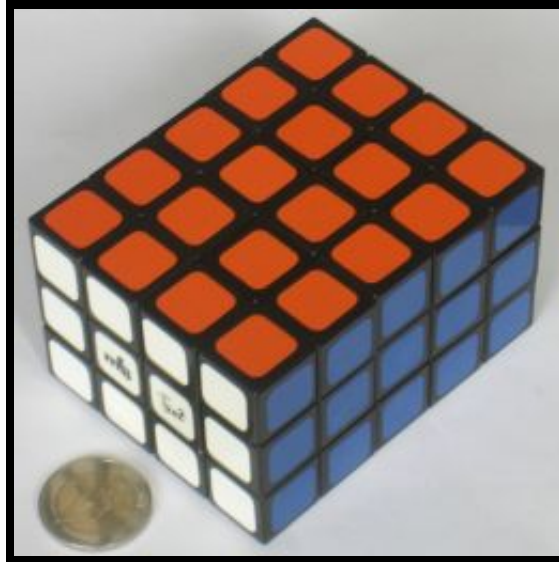# 3×4×5

This puzzle is like a 3×4×5 Rubik's cube. Any layer can be turned, but since they are all rectangular, only half turns will keep the puzzle in a cuboid shape.

If you do allow it to change shape by using quarter turns of the sides, then the centre of the puzzle acts like a 3×3×4 cuboid. The parts from the two outer 3×4 layers cannot be separated from its adjacent part until the shape has been changed so as to form a complete outer layer.

The number of positions:

First let's assume no shape-changing takes place.
Every movable piece can be moved to only four locations using half turns. There are 52 movable pieces (the centre facelet of a 3×5 faces does not move), which therefore split into 13 orbits of four pieces each. This gives a maximum of $4!^{13}$ possible positions, but this maximum is not reached because:

- The 5 orbits of face centres each have two pairs of indistinguishable pieces ($2!^{10}$)
- In the pair of corner orbits and the two pairs of wing edge orbits, the permutation of one orbit determines the relative locations of the pieces in the other orbit. ($3!^3$)
- The orientation of the puzzle as a whole does not matter (2)

The second condition essentially means that the location of five of the pieces in the two orbits determines the locations of the other three. This gives a total of $4!^{13} / (2!^{10} \cdot 3!^3 \cdot 2) = 1,981,355,655,168$ positions.

If we do allow the shape to be changed, then there are many more positions. The puzzle then consists of a 3×3×4 core, onto which the pieces of the outer layers are attached. The core puzzle stays the same shape, it is just the outer layer pieces that define the changing shape. First consider just the 3×3×4 core of the puzzle. This cuboid has 8 corners, 8 top/bottom edges, 8 edges in the middle layers, and 8 centres. This gives at most $8!^4$ positions,

but this maximum is not reached because:

- There are 4 pairs of indistinguishable centres ($2!^4$)
- The core's top/bottom edge pieces that show only one colour occur in 2 identical pairs ($2!^2$)
- The orientation of the puzzle as a whole does not matter (4)

The outer layer extensions consist of two sets of 4 corner pieces, two sets of 4 wing edges, and a set of 4 normal edges. The outer layer centres have already been taken into account above as part of the core puzzle. The parity constraints on the permutation of the outer edge pieces are immaterial since the base pieces that they are connected to come in identical pairs that can be swapped. The corner pieces however can only be mixed by an even permutation. So there are $4!^5/2$ possible permutations of these outer pieces, seemingly without affecting the core cuboid.

Putting it all together, the total number of positions is $8!^4 \cdot 4!^5 / (2!^4 \cdot 2!^2 \cdot 4 \cdot 2) = 41,102,509,778,424,299,520,000 = 4.11 \cdot 10^{22}$.

Notation:

Hold the puzzle with a 3×5 face on top. This face has a fixed face centre that can be used as a reference point. Assuming the puzzle is in a cuboid shape, hold a 4×5 face at the front, leaving the left and right faces as 3×4. If it is not in cuboid shape, then simply keep in mind that you have a vertical core 3×3×4, and that the outer pieces will eventually form extra layers on the left and right faces.

Clockwise quarter turns of the top or bottom layer will be denoted by U and D respectively, half turns by U2 and D2, and finally counter-clockwise turns by U' and D'. Turns of the upper and lower inner layers will be denoted in the same way, except that the lower case letters u and d are used. For the side faces the letters F, B, L, and R represent half turns of the front, back, left and right face respectively. Similarly, lower case letters l and r represent half turns of the vertical inner layers adjacent to the L and R face layers.

Solution:

Phase 1: Put the puzzle into cuboid shape.
This phase is obviously not necessary if you have not used quarter turns when scrambling the puzzle.

- Corners
  a. If three or more of the U layer corners stick out to the front or back, then do U.
  b. If three or more of the D layer corners stick out to the front or back, then do D.
  c. If you still have four U/D layer corners sticking out to the front or back, then use half turns to bring at least three of them to U layer, and then do U.
  d. If you still have two U/D layer corners sticking out to the front or back, then use half turns to bring them both to U layer. Turn the U layer so that so that you have two corners sticking out to the front only. Finally, do r U' B U B U to fix the corners.
- Top/Bottom edges.
  There are two kinds of top/bottom edges: Extended edges, that belong on the left or right, and normal edges, which belong on the front and back.
  e. If there is a top/bottom extended edge that sticks out to the front or back, then use half turns to bring that edge to the top back location.
  f. Find a normal top/bottom edge that lies on the left or right hand side. Use half turns to bring that edge to the top left location without disturbing the top-back extended edge (so only use l, r, and D2).
  g. Do l D' l F r U to swap the two edges.
  h. Repeat steps e-g until the top/bottom layers are rectangular.
- Middle layer edges.
  This is essentially the same as steps a-d for the corners.
  i. If three or more of the u layer edges stick out to the front or back, then do u.
  j. If three or more of the d layer edges stick out to the front or back, then do d.

    k. If you still have four u/d layer edges sticking out to the front or back, then use half turns to bring at least three of them to u layer, and then do u.

    l. If you still have two u/d layer edges sticking out to the front or back, then use half turns to bring them both to u layer. Turn the u layer so that so that you have two edges sticking out to the front only. Finally, do r u' B u B u to fix the edges.

- Middle layer centres.
  This is essentially the same as steps e-h for the top/bottom edges.
  There are two kinds of centres: Extended centres, that belong on the left or right, and normal centres, which belong on the front and back.

    e. If there is an extended centre that sticks out to the front or back, then use half turns to bring that centre to the top back location.

    f. Find a normal centre that lies on the left or right hand side. Use half turns to bring that centre to the top left location without disturbing the top-front extended centre (so only use l, r, and d2).

    g. Do l d' l F r u to swap the two centres.

    h. Repeat steps m-o until the middle layers are rectangular.

Phase 2: Solve the corners

a. Using half turns only, make sure that the back left corner of the top face matches the middle centre of the top face.

b. Using half turns only and without disturbing the first corner (i.e. using only F, R, D2), make sure that the front left corner of the top face matches the middle centre of the top face.

c. Using half turns only and without disturbing the first two corners (i.e. using only R, D2), make sure that the back right corner of the top face matches the middle centre of the top face.

d. If the front right corner does not match the middle centre (which can only happen if the puzzle was scrambled with quarter turns) then
    1. Find the fourth corner that has the colour of the top face centre.
    2. Bring that corner to the front left of the bottom face, if necessary, by doing a D2 move.
    3. Do F rR D' rR U F U2 rR D rR U to insert the corner into the top face.

e. If the side colours of the top four corners do not match each other, do R F R.

f. If the side colours of the bottom four corners do not match each other, do rR U2 F U rR F U2 F U F U2 rR U.

g. If necessary, do U2 to make the sides of the top layer corners match the bottom layer corners.

Phase 3: Solve the wing edges of the U and D layers.

a. Find the wing edge that belongs at the back left of the top layer (at UBl). Depending on where it is, do one of the following to solve it:
   UBl, back-left of top layer: do nothing.
   UFr, front-right of top layer: r U2 r U2.
   DFl, front-left of bottom layer: l.
   DBr, back-right of bottom layer: D2 l D2.

b. Find the wing edge that belongs at the front left of the top layer (at UFl). Depending on where it is, do one of the following to solve it:
   UFl, front-left of top layer: do nothing.
   UBr, back-right of top layer: F r F.
   DFr, front-right of bottom layer: r F r F.
   DBl, back-left of bottom layer: D2 r F r F D2.

c. Find the wing edge that belongs at the back right of the top layer (at UBr). Depending on where it is, do one of the following to solve it:
   UBr, back-right of top layer: do nothing.
   DBl, back-left of bottom layer: r.
   DFr, front-right of bottom layer: D2 r D2.

d. Find the wing edge that belongs at the front right of the top layer (at UFr). If it is not already in position

(which can only happen if the puzzle was scrambled with quarter turns) then do one of the following to solve it:
DBr, back-right of bottom layer: lL D rR U' lL U rR D' r D rR U' lL U rR D' lL r.
DFl, front-left of bottom layer: D2, followed by the sequence above, and then D2 again.

e. If the bottom layer wing edges are not yet correct, then do r D2 r D2 r.

Phase 4: Solve the remaining edges of the U and D layers.

a. Find the edge belonging at the back of the bottom layer (at DB). Depending on where it is, do one of the following to solve it:
DB, back of bottom layer: do nothing.
DF, front of bottom layer: D2 rR D2 rR D2 rR.
UB, back of top layer: B rR lL F rR lL.
UF, front of top layer: U2 B rR lL F rR lL U2.

b. Find the edge belonging at the front of the bottom layer (at DF). Depending on where it is, do one of the following to solve it:
DF, front of bottom layer: do nothing.
UB, back of top layer: rR F rR U2 rR F rR U2.
UF, front of top layer: U2 rR F rR U2 rR F rR.

c. If the two edges at the front and back of the top layer need to be swapped, then do U2 rR U2 rR U2 rR.

d. Find the edge belonging at the left of the bottom layer (at DL). Depending on where it is, do one of the following to solve it:
DL, left of bottom layer: do nothing.
DR, right of bottom layer: D2 F D2 F D2 F.
UL, left of top layer: lL F B rR F B.
UR, right of top layer: U2 lL F B rR F B U2.

e. Find the edge belonging at the right of the bottom layer (at DR). Depending on where it is, do one of the following to solve it:
DR, right of bottom layer: do nothing.
UL, left of top layer: F rR F U2 F rR F U2.
UR, right of top layer: U2 F rR F U2 F rR F.

f. If the two edges at the left and right of the top layer need to be swapped, then do U2 F U2 F U2 F.

Phase 5: Solve the edges of the middle layers.

a. Find the edge that belongs at the back left of the top inner layer (at uBL). Depending on where it is, do one of the following to solve it:
uBL, back-left of top inner layer: do nothing.
uFR, front-right of top inner layer: u2
dFL, front-left of bottom inner layer: F u2 F.
dBR, back-right of bottom inner layer: L d2 L.

b. Find the edge that belongs at the front left of the top inner layer (at uFL). Depending on where it is, do one of the following to solve it:
uFL, front-left of top inner layer: do nothing.
dBL, back-left of bottom inner layer: F d2 F.
dFR, front-right of bottom inner layer: d2 F d2 F.
uBR, back-right of top inner layer: R d2 R F d2 F.

c. Find the edge that belongs at the back right of the top inner layer (at uBR). Depending on where it is, do one of the following to solve it:
uBR, back-right of top inner layer: do nothing.
dBL, back-left of bottom inner layer: R d2 R.
dFR, front-right of bottom inner layer: d2 R d2 R.

d. Find the edge that belongs at the front right of the top inner layer (at uFR). If it is not already in position (which can only happen if the puzzle was scrambled with quarter turns) then do one of the following to

solve it:

dFL, front-left of bottom inner layer: F rR d' rR u F u2 rR d rR u

dBR, back-right of bottom inner layer: d2, followed by the sequence above, and then d2 again.

e. If the front right edge of the bottom inner layer is not correct, do d2.

f. If the bottom inner layer's edges are not yet correct, then do F rR u2 rR u F u2 rR u F u2 F rR u.

Phase 6: Solve the centres of the U and D faces.

a. If both the left and right centres of the top face (i.e. Ul and Ur) are not the correct colour, do r2 F B l2 F B.

b. Hold the puzzle such that the left centre of the bottom face is correct.

c. If the left centre of the top face (i.e. Ul) is not the correct colour, do F r F U2 F r F U2.

d. If the right centre of the top face (i.e. Ur) is not the correct colour, do U2 F r F U2 F r F.

Phase 7: Solve the centres of the R and L faces.

a. If both centres of the left face (i.e. uL and dL) are not the correct colour, do u2 F B d2 F B.

b. Hold the puzzle such that the bottom centre of the left face is correct.

c. If the top centre of the right face (i.e. uR) is not the correct colour, do R F u2 F R F u2 F.

d. If the bottom centre of the right face (i.e. dR) is not the correct colour, do F u2 F R F u2 F R.

Phase 8: Solve the front and back centres.

a. If both middle centres of the front face (i.e. uF and dF) are not the correct colour, do u2 rR lL d2 rR lL.

b. Hold the puzzle such that the bottom middle centre of the back face is correct.

c. If the top middle centre of the front face (i.e. uF) is not the correct colour, do F rR u2 rR F rR u2 rR.

d. If the bottom middle centre of the front face (i.e. dF) is not the correct colour, do rR u2 rR F rR u2 rR F.

e. If the top-left centre of the front face (i.e. Ful) is not the correct colour, do r2 u2 r2 u2 once or twice until it is.

f. If the bottom-right centre of the front face (i.e. Fdr) is not the correct colour, do r2 d2 r2 d2 once or twice until it is.

g. If the top-right and bottom-left centres of the front face (i.e. Fur and Fdl) are both not the correct colour, do F r2 u2 r2 u2 F r2 u2 r2 u2.

h. If there are still two incorrect centres left (which can only happen if the puzzle was scrambled with quarter turns) then hold the puzzle so that one of those centres lies at the bottom-right of the back face (at Bdr), and do one of the following depending on which of the front centres is incorrect:

Fdl, bottom-left centre of front face: lL d rR u' lL u rR d' r d rR u' lL u rR d' lL r.

Fur, top-right centre of front face: F, followed by the sequence above, and then F again.

Home	**Links**	**Guestbook**