HW3: CS29a/Aut08

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due Mon 10/13

October 6, 2008

1 Binary Relations

Let $S = \{a, b, c, d, e, f, g\}$ be a set of elements and let R be the relation defined by the following subset of $A \times A$

$$R = \{(a, b), (b, c), (c, d), (d, e), (e, f), (c, a), (c, g), (e, d)\}$$

Let R^* be the transitive closure of R and define the binary relation E on A by

$$aEb \Leftrightarrow (aR^*b \wedge bR^*a) \vee (a = b)$$

1. Draw the digraph corresponding to R
2. Write down the 7x7 matrix corresponding to R
3. Express R as a multi-valued partial function from A to A
4. Explain why E is an equivalence relations (reflexive, symmetrix, transitive)
5. Let A' be the set of equivalence classes on E. Write down the elements of A' (each one will be a subset of A)
6. Define a relation S on A' by

   $$a'Sb' \Leftrightarrow \exists a \in a', b \in b'.aRb$$

   and draw the relation S as a digraph on A'

2 POsets

Let $A = \{a, b, c, d, e, f\}$ be a set and let R be the binary relation on $A \times A$ defined by

$$R = \{(b, c), (b, d), (c, e), (d, e), (a, d), (c, f), (d, f)\}$$

Define R^2 by

$$aR^2b \Leftrightarrow \exists c.aRc \wedge cRb$$

Define the function g from A to the powerset $P(A)$ by

$$g(x) = \{x\} \cup \{y|yR^*x\}$$

that is $g(x)$ is the set of all nodes in A for which there is a path from y to x.

1. Draw R as a digraph.
2. Draw R^2 as a digraph.
3. Is R a DAG? Why or why not?
4. Let R^* be the transitive closure of R. What new edges do we need to add to R to form the transitive closure.
5. Make a table that shows $g(x)$ for each x in A.
6. Verify that $g(c) \subset g(b)$ but that $g(c)$ and $g(d)$ are incomparable (i.e. neither contains the other).

3 Logic Relations

Let A denote the set consisting of the following six formulas:

a_1: $\forall x \forall y. L(x, y)$

a_2: $\forall x \exists y. L(x, y)$

a_3: $\exists x \forall y. L(x, y)$

a_4: $\forall y \exists x. L(x, y)$

a_5: $\exists y \forall x. L(x, y)$

a_6: $\exists x \exists y. L(x, y)$

Define a relation R on A by

$$aRb \Leftrightarrow (a \rightarrow b)$$

that is aRb is true precisely when a implies b as logical formulas. For example, $a_1 R a_6$ because if everyone loves everyone then there must be someone that loves someone (possibly themselves), but the converse is not true. Just because one person loves another doesn't mean everybody must love everyone.

- Give an English translation for each of the six formulas in A. For example, a_4 could be translated as "Everyone is loved by at least one person"
- Draw the relation R as a digraph and explain why your graph is correct (e.g. why do we have $a_3 \rightarrow a_4$ but we don't have $a_4 \rightarrow a_3$?)

4 Models of Logical Formulas

Let A denote the six formulas from the previous problem and let U be a set of people and assume that $L(x, y)$ means that x loves y where x, y are in U. Assume also that it is unambiguously known for each pair of people a, b in U if a loves b or not.

Suppose that $U = \{u, v\}$ contains just two people (uma and vlad) then the predicate L can be interpreted as a binary relation on U and hence as a digraph on U.

When a particular choice is made for L, the pair (U, L) is called an interpretation for a logical formulas containing L, and it is a model for such a formula if that formula is true for that particular interpretation.

- Find a relation L that makes a_4 true, but not a_3. This is a counterexample to the claim that $a_4 \to a_3$.
- Find a relation L that makes a_4 true, but not a_2. This is a counterexample to the claim that $a_4 \to a_2$.
- How many ways are there to define a binary relation L on U
- Draw all of the interpretation of L on $\{u, v\}$ for which a_4 is true.