CS112 Notes
Modal Logic: Part I

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1 Modal Logic

So far in this class, we have studied propositional and first-order systems of representation and reasoning. The functions we assume to valuate the truth of a proposition in either system have involved assignment within a model, to a value of either true or false.

Modal logic is a form of reasoning that allows us to qualify the truth of a judgment in our system. Truth is assigned to a proposition in a *model*, where a model is a structured set of worlds, where each world is labeled with truth assignments to propositions. The *structure* given to the set of worlds is a relation called the *Accessibility Relation*, R. We will define these more precisely, shortly. Modal logic studies reasoning that involves the use of expressions such as:

It is necessary/possible that ..
It is obligatory/permitted/forbidden that ..
It will always be the case that ..
It will be the case that ..
It has always been the case that ..
It was the case that..
x believes that ..

1.1 Systems of Modal Logic

The system of *classical modal logic* called K (named after Kripke) results from propositional logic with the addition of the following rule schema and axiom:

(a) **Necessitation Rule:** If ϕ is a theorem of K, then so is $\Box\phi$.

(b) **Distribution Axiom:** $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

The possibility operator $\Diamond$ can be defined from $\Box$ with the following equivalence:

$$\Diamond\phi \;=\; \neg\Box\neg\phi.$$

With the addition of the axiom below, we arrive at a proper modal logic, called system T.

(c) (T) $\Box\phi \rightarrow \phi$

From system T ((a-c)), the system *S4* is defined as including still another axiom, given below:

(d) (4) $\Box A \rightarrow \Box\Box A$

In S4, the behavior of stacked operators given the rule in (d) is as follows:

(i) $\Box\Box \ldots \Box A \;\vdash\; \Box A$

(ii) $\Diamond\Diamond \ldots \Diamond A \;\vdash\; \Diamond A$

So, for example, $\Box\Box A \vdash \Box A$. To show this, we take a proof by refutation, where Γ is $\neg\Box A$.

(1) $\Box\Box A$ by Δ.

(2) $\neg\Box A$ by Γ.

(3) $\Box A$ by rule (c).

(4) $\perp$ 2,3.

When there are stacked $\Diamond$'s, then rule (d) gets invoked. Let us sketch the proof of $\Diamond\Diamond\Diamond A \vdash \Diamond A$.

(1) $\Diamond\Diamond\Diamond A$ by Δ.

(2) $\neg\Diamond A$ by Γ.

(3) $\Box\neg A$ by equivalence on (2).

(4) $\Box\Box\neg A$ by rule (d) on (3).

(5) $\Box\Box\Box\neg A$ by rule (d) on (4).

(6) $\neg\Box\Box\Box\neg A$ by equivalence on (1).

(7) $\perp$ by 5,6.

From (a)-(c), another system, *S5* can be defined with one additional axiom:

(e) (5) $\Diamond A \rightarrow \Box\Diamond A$

The difference here is that it doesn't matter what operators are stacked. The determining operator is the one locally dominating the proposition, A. The behavior of stacked operators given the rule (e) is as follows:

(i) $\mathcal{O}\mathcal{O} \ldots \Box A \;\vdash\; \Box A$

(ii) $\mathcal{O}\mathcal{O} \ldots \Diamond A \;\vdash\; \Diamond A$

For example, to prove that $\Diamond\Box\Box A$ reduces to $\Box A$, we proceed as follows. First, we prove that $\Diamond\Box\Box A \vdash \Box\Box A$.

(1) $\Diamond\Box\Box A$ by Δ.

(2) $\neg\Box\Box A$ by Γ.

(3) $\Diamond\Diamond\neg A$ by equivalence on (2).

(4) $\Box\Diamond\Diamond\neg A$ by rule (e) on (3).

(5) $\neg\Box\Diamond\Diamond\neg A$ by equivalence on (1).

(6) $\perp$ by 4, 5.

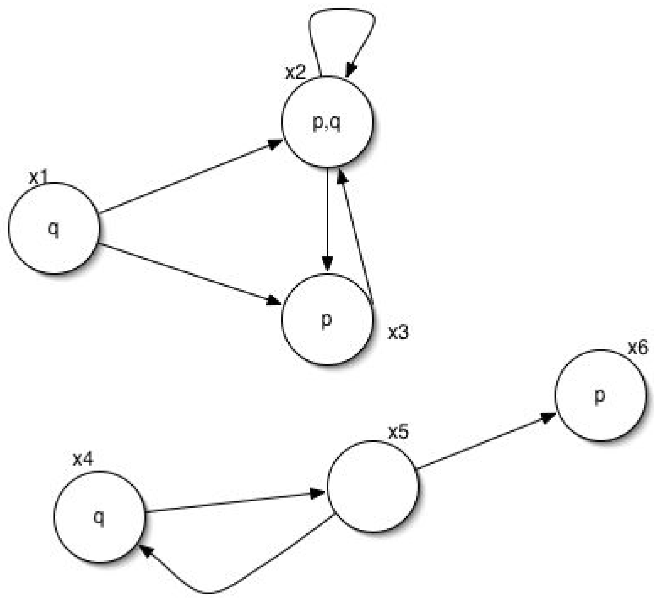
Then, we prove that $\Box\Box A \;\vdash\; \Box A$. This follows directly from (c) above. Hence, $\Diamond\Box\Box A \;\vdash\; \Box A$. Principle (c) is invoked in S5 when there are stacked boxes.

1.2 Example Kripke Model and Satisfiability Properties

Consider a model $\mathcal{M} = (W, R, L)$ of basic modal logic. $\mathcal{M}$ is said to satisfy a formula if every state in the model satisfies it. That is,

$\mathcal{M} \vDash \phi$ iff for each $x \in W$, $x \Vdash \phi$.

For example, in the model below, we can find the following satisfaction properties:



a. $x_1 \Vdash q$, since $q \in L(x_1)$.

b. $x_1 \Vdash \Diamond q$, for there is a world related to x_1 (i.e., x_2) which satisfies q. That is, $R(x_1, x_2)$ and $x_2 \Vdash q$.

c. $x_5 \nVdash \Box p$ and $x_5 \nVdash \Box q$. Moreover, $x_5 \nVdash \Box p \vee \Box q$. But notice that $x_5 \Vdash \Box(p \vee q)$. Notice that the worlds relating to x_5 are x_4 and x_6. Since $x_4 \nVdash p$, we have $x_5 \nVdash \Box p$. And, since $x_6 \nVdash q$, we have $x_5 \nVdash \Box q$. Hence, we get $x_5 \nVdash \Box p \vee \Box q$. But $x_5 \Vdash \Box(p \vee q)$ because in each of x_4 and x_6, we find p or q.

1.3 Properties of Accessibility Relation, R, in PML

$Name$	$Formula\ Scheme$	$Property\ of\ R$
T	$\Box\phi \rightarrow \phi$	$reflexive$
B	$\phi \rightarrow \Box\Diamond\phi$	$symmetric$
D	$\Box\phi \rightarrow \Diamond\phi$	$serial$
4	$\Box\phi \rightarrow \Box\Box\phi$	$Transitive$
5	$\Diamond\phi \rightarrow \Box\Diamond\phi$	$Euclidean$

1.4 Natural Deduction for Modal Logic

Proving validity in modal logic with natural deduction is very similar to what we presented for propositional logic. The major difference is that a new kind of proof box is introduced which has *dashed* lines, for invoking elimination and introduction of the connective $\Box$. The dashed box can be interpreted as reasoning in an *arbitrary related world* (rather than object).

(i) $\Box$-i: If at any point in a proof we have $\Box\phi$, we can open a dashed box and put ϕ in it. From this, we can obtain, for example, ψ. Then we come out of the dashed box, and since we have shown ψ in an arbitrary world, we can conclude $\Box\psi$ outside the dashed box.

(ii) $\Box$-e: If $\Box\phi$ occurs somewhere in a proof, ϕ may be put into a subsequent dashed box for reasoning in an arbitrary world.

2 Modal Predicate Logic

2.1 Models for Modal Predicate Logic

A model **M** for a modal predicate logical language L consists of:

(i) a nonempty set W of possible worlds

(ii) an accessibility relation R on W

(iii) a domain function D which assigns a domain D_w to each worlds $w \in W$

(iv) an interpretation function I which assigns an entity $I(c)$ to each constant c of L, and for every world $w \in W$ a subset $I_w(P)$ of $(D_w)^n$ to each n-ary predicate letter P of L.

Let **M** be a model, $w \in W$, and ϕ a formula of modal predicate logic without variables. $V_{M,w}(\phi)$, the truth value of ϕ in w given **M** is defined as follows:

(a) $V_{M,w}(P(c_1, \ldots, c_n)) = 1$ iff $I(c_i) \in D_w$, and $I(c_i) \in I_w(P)$ for each c_i
$V_{M,w}(P(c_1, \ldots, c_n)) = 1$ iff $I(c_i) \in D_w$, and $I(c_i) \notin I_w(P)$ for each c_i

(b) $V_{M,w}(\neg\phi) = 1$ iff $V_{M,w}(\phi) = 0$
$V_{M,w}(\neg\phi) = 0$ iff $V_{M,w}(\phi) = 1$

(c) $V_{M,w}(\phi \to \psi) = 0$ iff $V_{M,w}(\phi) = 1$ and $V_{M,w}(\psi) = 0$
$V_{M,w}(\phi \to \psi) = 1$ iff $V_{M,w}(\phi) = 1$ and $V_{M,w}(\psi) = 1$
or $V_{M,w}(\phi) = 0$ and $V_{M,w}(\psi) = 1$
or $V_{M,w}(\phi) = 0$ and $V_{M,w}(\psi) = 0$

(d) $V_{M,w}(\Box\phi) = 1$ iff for every $w_i \in W$ such that wRw_i: $V_{M,w_i}(\phi) = 1$
$V_{M,w}(\Box\phi) = 0$ iff there is a $w_i \in W$ such that wRw_i: $V_{M,w_i}(\phi) = 0$

References

Huth, M. and M. Ryan (2000) *Logic in Computer Science: Modelling and Reasoning about Systems*, Cambridge University Press, Cambridge.