PROOFS AND TYPES

JEAN-YVES GIRARD

Translated and with appendices by

PAUL TAYLOR

YVES LAFONT

CAMBRIDGE UNIVERSITY PRESS

Cambridge

New York *New Rochelle*

Melbourne *Sydney*

Chapter 5

Sequent Calculus

The *sequent calculus*, due to Gentzen, is the prettiest illustration of the symmetries of Logic. It presents numerous analogies with natural deduction, without being limited to the intuitionistic case.

This calculus is generally ignored by computer scientists[1]. Yet it underlies essential ideas: for example, `PROLOG` is an implementation of a fragment of sequent calculus, and the "tableaux" used in automatic theorem-proving are just a special case of this calculus. In other words, it is used unwittingly by many people, but mixed with *control* features, *i.e.* programming devices. What makes everything work is the sequent calculus with its deep symmetries, and not particular tricks. So it is difficult to consider, say, the theory of `PROLOG` without knowing thoroughly the subtleties of sequent calculus.

From an algorithmic viewpoint, the sequent calculus has no *Curry-Howard isomorphism*, because of the multitude of ways of writing the same proof. This prevents us from using it as a typed λ-calculus, although we glimpse some deep structure of this kind, probably linked with parallelism. But it requires a new approach to the syntax, for example natural deductions with several conclusions.

[1] An exception is [Gallier].

5.1 The calculus

5.1.1 Sequents

A *sequent* is an expression $\underline{A} \vdash \underline{B}$ where $\underline{A}$ and $\underline{B}$ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$.

The naïve (denotational) interpretation is that the conjunction of the A_i implies the disjunction of the B_j. In particular,

- if $\underline{A}$ is empty, the sequent asserts the disjunction of the B_j;
- if $\underline{A}$ is empty and $\underline{B}$ is just B_1, it asserts B_1;
- if $\underline{B}$ is empty, it asserts the negation of the conjunction of the A_i;
- if $\underline{A}$ and $\underline{B}$ are empty, it asserts contradiction.

5.1.2 Structural rules

These rules, which seem not to say anything at all, impose a certain way of managing the "slots" in which one writes formulae. They are:

1. The *exchange* rules

$$\frac{\underline{A}, C, D, \underline{A'} \vdash \underline{B}}{\underline{A}, D, C, \underline{A'} \vdash \underline{B}}\ \mathcal{L}\mathsf{X} \qquad\qquad \frac{\underline{A} \vdash \underline{B}, C, D, \underline{B'}}{\underline{A} \vdash \underline{B}, D, C, \underline{B'}}\ \mathcal{R}\mathsf{X}$$

   These rules express in some way the *commutativity* of logic, by allowing permutation of formulae on either side of the symbol "$\vdash$".

2. The *weakening* rules

$$\frac{\underline{A} \vdash \underline{B}}{\underline{A}, C \vdash \underline{B}}\ \mathcal{L}\mathsf{W} \qquad\qquad \frac{\underline{A} \vdash \underline{B}}{\underline{A} \vdash C, \underline{B}}\ \mathcal{R}\mathsf{W}$$

   as their name suggests, allow replacement of a sequent by a weaker one.

3. The *contraction* rules

$$\frac{\underline{A}, C, C \vdash \underline{B}}{\underline{A}, C \vdash \underline{B}}\ \mathcal{L}\mathsf{C} \qquad\qquad \frac{\underline{A} \vdash C, C, \underline{B}}{\underline{A} \vdash C, \underline{B}}\ \mathcal{R}\mathsf{C}$$

   express the idempotence of conjunction and disjunction.

In fact, contrary to popular belief, these rules are the most important of the whole calculus, for, without having written a single logical symbol, we have practically determined the future behaviour of the logical operations. Yet these rules, if they are obvious from the denotational point of view, should be examined closely from the operational point of view, especially the *contraction*.

It is possible to envisage variants on the sequent calculus, in which these rules are abolished or extremely restricted. That seems to have some very beneficial effects, leading to linear logic [Gir87]. But without going that far, certain well-known restrictions on the sequent calculus seem to have no purpose apart from controlling the structural rules, as we shall see in the following sections.

5.1.3 The intuitionistic case

Essentially, the intuitionistic sequent calculus is obtained by restricting the form of sequents: an *intuitionistic sequent* is a sequent $\underline{A} \vdash \underline{B}$ where $\underline{B}$ is a sequence formed from *at most one* formula. In the intuitionistic sequent calculus, the only structural rule on the right is $\mathcal{R}\mathsf{W}$ since $\mathcal{R}\mathsf{X}$ and $\mathcal{R}\mathsf{C}$ assume several formulae on the right.

The intuitionistic restriction is in fact a modification to the management of the formulae — the particular place distinguished by the symbol $\vdash$ is a place where contraction is forbidden — and from that, numerous properties follow. On the other hand, this choice is made at the expense of the left/right symmetry. A better result is without doubt obtained by forbidding contraction (and weakening) altogether, which allows the symmetry to reappear.

Otherwise, the intuitionistic sequent calculus will be obtained by restricting to the intuitionistic sequents, and preserving — apart from one exception — the classical rules of the calculus.

5.1.4 The "identity" group

1. For every formula C there is the *identity axiom* $C \vdash C$. In fact one could limit it to the case of atomic C, but this is rarely done.

2. The *cut rule*

$$\frac{\underline{A} \vdash C, \underline{B} \qquad \underline{A'}, C \vdash \underline{B'}}{\underline{A}, \underline{A'} \vdash \underline{B}, \underline{B'}}\ \mathsf{Cut}$$

   is another way of expressing the identity. The identity axiom says that C (on the left) is stronger than C (on the right); this rule states the converse truth, *i.e.* C (on the right) is stronger than C (on the left).

The identity axiom is absolutely necessary to any proof, to start things off. That is undoubtedly why the cut rule, which represents the dual, symmetric aspect can be eliminated, by means of a difficult theorem (proved in chapter 13) which is related to the normalisation theorem. The deep content of the two results is the same; they only differ in their syntactic dressing.

5.1.5 Logical rules

There is tradition which would have it that Logic is a formal game, a succession of more or less arbitrary axioms and rules. Sequent calculus (and natural deduction as well) shows this is not at all so: one can amuse oneself by inventing one's own logical operations, but they have to respect the left/right symmetry, otherwise one creates a logical atrocity without interest. Concretely, the symmetry is the fact that we can *eliminate* the cut rule.

1. *Negation*: the rules for negation allow us to pass from the right hand side of "$\vdash$" to the left, and conversely:

$$\frac{\underline{A} \vdash C, \underline{B}}{\underline{A}, \neg C \vdash \underline{B}}\,\mathcal{L}\neg \qquad\qquad \frac{\underline{A}, C \vdash \underline{B}}{\underline{A} \vdash \neg C, \underline{B}}\,\mathcal{R}\neg$$

2. *Conjunction*: on the left, two unary rules; on the right, one binary rule:

$$\frac{\underline{A}, C \vdash \underline{B}}{\underline{A}, C \wedge D \vdash \underline{B}}\,\mathcal{L}1\wedge \qquad\qquad \frac{\underline{A}, D \vdash \underline{B}}{\underline{A}, C \wedge D \vdash \underline{B}}\,\mathcal{L}2\wedge$$

$$\frac{\underline{A} \vdash C, \underline{B} \qquad \underline{A'} \vdash D, \underline{B'}}{\underline{A}, \underline{A'} \vdash C \wedge D, \underline{B}, \underline{B'}}\,\mathcal{R}\wedge$$

3. *Disjunction*: obtained from conjunction by interchanging right and left:

$$\frac{\underline{A}, C \vdash \underline{B} \qquad \underline{A'}, D \vdash \underline{B'}}{\underline{A}, \underline{A'}, C \vee D \vdash \underline{B}, \underline{B'}}\,\mathcal{L}\vee$$

$$\frac{\underline{A} \vdash C, \underline{B}}{\underline{A} \vdash C \vee D, \underline{B}}\,\mathcal{R}1\vee \qquad\qquad \frac{\underline{A} \vdash D, \underline{B}}{\underline{A} \vdash C \vee D, \underline{B}}\,\mathcal{R}2\vee$$

Special case: The intuitionistic rule $\mathcal{L}\vee$ is written:

$$\frac{\underline{A}, C \vdash \underline{B} \quad \underline{A'}, D \vdash \underline{B}}{\underline{A}, \underline{A'}, C \vee D \vdash \underline{B}} \mathcal{L}\vee$$

where $\underline{B}$ contains zero or one formula. This rule is not a special case of its classical analogue, since a classical $\mathcal{L}\vee$ leads to $\underline{B}, \underline{B}$ on the right. This is the only case where the intuitionistic rule is not simply a restriction of the classical one.

4. *Implication*: here we have on the left a rule with two premises and on the right a rule with one premise. They match again, but in a different way from the case of conjunction: the rule with one premise uses *two* occurrences in the premise:

$$\frac{\underline{A} \vdash C, \underline{B} \quad \underline{A'}, D \vdash \underline{B'}}{\underline{A}, \underline{A'}, C \Rightarrow D \vdash \underline{B}, \underline{B'}} \mathcal{L}\Rightarrow \qquad \frac{\underline{A}, C \vdash D, \underline{B}}{\underline{A} \vdash C \Rightarrow D, \underline{B}} \mathcal{R}\Rightarrow$$

5. *Universal quantification*: two unary rules which match in the sense that one uses a *variable* and the other a *term*:

$$\frac{\underline{A}, C[a/\xi] \vdash \underline{B}}{\underline{A}, \forall \xi.\, C \vdash \underline{B}} \mathcal{L}\forall \qquad \frac{\underline{A} \vdash C, \underline{B}}{\underline{A} \vdash \forall \xi.\, C, \underline{B}} \mathcal{R}\forall$$

$\mathcal{R}\forall$ is subject to a restriction: ξ must not be free in $\underline{A}, \underline{B}$.

6. *Existential quantification*: the mirror image of 5:

$$\frac{\underline{A}, C \vdash \underline{B}}{\underline{A}, \exists \xi.\, C \vdash \underline{B}} \mathcal{L}\exists \qquad \frac{\underline{A} \vdash C[a/\xi], \underline{B}}{\underline{A} \vdash \exists \xi.\, C, \underline{B}} \mathcal{R}\exists$$

$\mathcal{L}\exists$ is subject to the same restriction as $\mathcal{R}\forall$: ξ must not be free in $\underline{A}, \underline{B}$.

5.2 Some properties of the system without cut

Gentzen's calculus is a possible formulation of first order logic. Gentzen's theorem, which is proved in chapter 13, says that the cut rule is redundant, superfluous. The proof is very delicate, and depends on the perfect right/left symmetry which we have seen. Let us be content with seeing some of the more spectacular consequences.

5.2.1 The last rule

If we can prove A in the predicate calculus, then it is possible to show the sequent $\vdash A$ *without cut*. What is the last rule used? Surely not $\mathcal{R}\mathsf{W}$, because the empty sequent is not provable. Perhaps it is the logical rule $\mathcal{R}i\mathsf{s}$ where s is the principal symbol of A, and this case is very important. But it may also be $\mathcal{R}\mathsf{C}$, in which case we are led to $\vdash A, A$ and all is lost! That is why the intuitionistic case, with its special management which forbids contraction on the right, is very important: if A is provable in the intuitionistic sequent calculus by a cut-free proof, then the last rule is a right logical rule.

Two particularly famous cases:

- If A is a disjunction $A' \vee A''$, the last rule must be $\mathcal{R}1\vee$, in which case $\vdash A'$ is provable, or $\mathcal{R}2\vee$, in which case $\vdash A''$ is provable: this is what is called the *Disjunction Property*.
- If A is an existence $\exists\xi.\,A'$, the last rule must be $\mathcal{R}1\exists$, which means that the premise is of the form $\vdash A'[a/\xi]$; in other words, a term t can be found such that $\vdash A'[a/\xi]$ is provable: this is the *Existence Property*.

These two examples fully justify the interest of limiting the use of the structural rules, a limitation which leads to linear logic.

5.2.2 Subformula property

Let us consider the last rule of a proof: can one somehow predict the premises? The cut rule is absolutely unpredictable, since an arbitrary formula C disappears: it cannot be recovered from the conclusions. It is the only rule which behaves so badly. Indeed, all the other rules have the property that the unspecified "context" part (written $\underline{A}$, $\underline{B}$, *etc.*) is preserved intact. The rule actually concerns only a few of the formulae. But the formulae in the premises are simpler than the corresponding ones in the conclusions. For example, for $A \wedge B$ as a conclusion, A and B must have been used as premises, or for $\forall\xi.\,A$ as a conclusion, $A[a/\xi]$ must have been used as a premise. In other words, one has to use *subformulae* as premises:

- The immediate subformulae of $A \wedge B$, $A \vee B$ and $A \Rightarrow B$ are A and B.
- The only immediate subformula of $\neg A$ is A.
- The immediate subformulae of $\forall\xi.\,A$ and $\exists\xi.\,A$ are the formulae $A[a/\xi]$ where a is any term.

Now it is clear that all the rules — except the cut — have the property that the premises are made up of subformulae of the conclusion. In particular, a cut-free proof of a sequent uses only subformulae of its formulae. We shall prove the corresponding result for natural deduction in section 10.3.1. This is very interesting for *automated deduction*. Of course, it is not enough to make the predicate calculus *decidable*, since we have an infinity of subformulae for the sentences with quantifiers.

5.2.3 Asymmetrical interpretation

We have described the identity axiom and the cut rule as the two faces of "A is A". Now, in the absence of cut, the situation is suddenly very different: we can no longer express that A (on the right) is stronger than A (on the left). Then there arises the possibility of splitting A into two interpretations $A^{\mathcal{L}}$ and $A^{\mathcal{R}}$, which need not necessarily coincide. Let us be more precise.

In a sentence, we can define the *signature* of an occurrence of an atomic predicate, $+1$ or -1: the signature is the parity of the number of times that this symbol has been negated. Concretely, P retains the signature which it had in A, when it is considered in $A \wedge B$, $B \wedge A$, $A \vee B$, $B \vee A$, $B \Rightarrow A$, $\forall \xi. A$ and $\exists \xi. A$, and reverses it in $\neg A$ and $A \Rightarrow B$.

In a sequent too, we can define the signature of an occurrence of a predicate: if P occurs in A on the left of "$\vdash$", the signature is reversed, if P occurs on the right, it is conserved.

The rules of the sequent calculus (apart from the identity axiom and the cut) preserve the signature: in other words, they relate occurrences with the same signature. The identity axiom says that the negative occurrences (signature -1) are stronger than the positive ones; the cut says the opposite. So in the absence of cut, there is the possibility of giving asymmetric interpretations to sequent calculus: A does not have the same meaning when it is on the right as when it is on the left of "$\vdash$".

- $A^{\mathcal{R}}$ is obtained by replacing the positive occurrences of the predicate P by $P^{\mathcal{R}}$ and the negative ones by $P^{\mathcal{L}}$.

- $A^{\mathcal{L}}$ is obtained by replacing the positive occurrences of the predicate P by $P^{\mathcal{L}}$ and the negative ones by $P^{\mathcal{R}}$.

The atomic symbols $P^{\mathcal{R}}$ and $P^{\mathcal{L}}$ are tied together by a condition, namely $P^{\mathcal{L}} \Rightarrow P^{\mathcal{R}}$.

It is easy to see that this kind of asymmetrical interpretation is consistent with the system without cut, interpreting $\underline{A} \vdash \underline{B}$ by $\underline{A}^{\mathcal{L}} \vdash \underline{B}^{\mathcal{R}}$.

The sequent calculus seems to lend itself to some much more subtle asymmetrical interpretations, especially in linear logic.

5.3 Sequent Calculus and Natural Deduction

We shall consider here the noble part of natural deduction, that is, the fragment without $\vee$, $\exists$ or $\neg$. We restrict ourselves to sequents of the form $\underline{A} \vdash B$; the correspondence with natural deduction is given as follows:

- To a proof of $\underline{A} \vdash B$ corresponds a deduction of B under the hypotheses, or rather parcels of hypotheses, $\underline{A}$.
- Conversely, a deduction of B under the (parcels of) hypotheses $\underline{A}$ can be represented in the sequent calculus, but unfortunately not uniquely.

From a proof of $\underline{A} \vdash B$, we build a deduction of B, of which the hypotheses are parcels, each parcel corresponding in a precise way to a formula of $\underline{A}$.

1. The axiom $A \vdash A$ becomes the deduction A .
2. If the last rule is a cut

$$\frac{\underline{A} \vdash B \quad \underline{A}', B \vdash C}{\underline{A}, \underline{A}' \vdash C}\ \mathsf{Cut}$$

and the deductions δ of $\begin{matrix}\underline{A}\\ \vdots\\ B\end{matrix}$ and δ' of $\begin{matrix}\underline{A}', B\\ \vdots\\ C\end{matrix}$ are associated to the sub-proofs above the two premises, then we associate to our proof the deduction δ' where all the occurrences of B in the parcel it represents are replaced by δ:

$$\begin{matrix} & \underline{A} \\ & \vdots \\ \underline{A}', & B \\ \vdots & \\ C & \end{matrix}$$

In general the hypotheses in the parcel in $\underline{A}$ are proliferated, but the number is preserved by putting in the same parcel afterwards the hypotheses which came from the same parcel before and have been duplicated. No regrouping occurs between $\underline{A}$ and $\underline{A}'$.

3. The rule $\mathcal{L}\mathsf{X}$

$$\frac{\underline{A}, C, D, \underline{A}' \vdash B}{\underline{A}, D, C, \underline{A}' \vdash B}\,\mathcal{L}\mathsf{X}$$

is interpreted as the identity: the same deduction before and after the rule.

4. The rule $\mathcal{L}\mathsf{W}$

$$\frac{\underline{A} \vdash B}{\underline{A}, C \vdash B}\,\mathcal{L}\mathsf{W}$$

is interpreted as the creation of a mock parcel formed from zero occurrences of C. Weakening is then the possibility of forming empty parcels.

5. The rule $\mathcal{L}\mathsf{C}$

$$\frac{\underline{A}, C, C \vdash B}{\underline{A}, C \vdash B}\,\mathcal{L}\mathsf{C}$$

is interpreted as the unification of two C-parcels into one. Contraction is then the possibility of forming big parcels.

6. The rule $\mathcal{R}\wedge$

$$\frac{\underline{A} \vdash B \qquad \underline{A}' \vdash C}{\underline{A}, \underline{A}' \vdash B \wedge C}\,\mathcal{R}\wedge$$

will be interpreted by $\wedge\mathcal{I}$: suppose that deductions ending in B and C have been constructed to represent the proofs above the two premises; then our proof is interpreted by:

$$\frac{\begin{array}{c}\underline{A}\\ \vdots\\ B\end{array} \qquad \begin{array}{c}\underline{A}'\\ \vdots\\ C\end{array}}{B \wedge C}\,\wedge\mathcal{I}$$

7. The rule $\mathcal{R}{\Rightarrow}$ will be interpreted by ${\Rightarrow}\mathcal{I}$:

$$\frac{\underline{A}, B \vdash C}{\underline{A} \vdash B \Rightarrow C}\,\mathcal{R}{\Rightarrow} \qquad \text{becomes} \qquad \frac{\begin{array}{c}\underline{A}, [B] \\ \vdots \\ C\end{array}}{B \Rightarrow C}\,{\Rightarrow}\mathcal{I}$$

   where a complete B-parcel is discharged at one go.

8. The rule $\mathcal{R}\forall$ will be interpreted by $\forall\mathcal{I}$:

$$\frac{\underline{A} \vdash B}{\underline{A} \vdash \forall\xi.\, B}\,\mathcal{R}\forall \qquad \text{becomes} \qquad \frac{\begin{array}{c}\underline{A} \\ \vdots \\ B\end{array}}{\forall\xi.\, B}\,\forall\mathcal{I}$$

9. With the left rules appears one of the hidden properties of natural deduction, namely that the elimination rules (which correspond *grosso modo* to the left rules of sequents) are written backwards! This is nowhere seen better than in linear logic, which makes the lost symmetries reappear. Here concretely, this is reflected in the fact that the left rules are translated by actions on parcels of hypotheses.

   The rule $\mathcal{L}1\wedge$ becomes $\wedge1\mathcal{E}$:

$$\frac{\underline{A}, B \vdash D}{\underline{A}, B \wedge C \vdash D}\,\mathcal{L}1\wedge \qquad \text{is interpreted by} \qquad \begin{array}{c}\underline{A}, \quad \dfrac{B \wedge C}{B}\,\wedge1\mathcal{E} \\ \vdots \\ D\end{array}$$

   $\wedge1\mathcal{E}$ allows us to pass from a $(B \wedge C)$-parcel to a B-parcel.

   Similarly, the rule $\mathcal{L}2\wedge$ becomes $\wedge2\mathcal{E}$.

10. The rule $\mathcal{L}{\Rightarrow}$ becomes ${\Rightarrow}\mathcal{E}$:

$$\frac{\underline{A} \vdash B \quad \underline{A}', C \vdash D}{\underline{A}, \underline{A}', B \Rightarrow C \vdash D}\,\mathcal{L}{\Rightarrow} \qquad \text{is interpreted by} \qquad \begin{array}{c} \underline{A}',\ \dfrac{\begin{array}{c}\underline{A}\\ \vdots \\ B\end{array} \quad B \Rightarrow C}{C}\,{\Rightarrow}\mathcal{E} \\ \vdots \\ D \end{array}$$

Here again, a C-parcel is replaced by a $(B \Rightarrow C)$-parcel; something must also be done about the proliferation of A-parcels, as in case 2.

11. Finally the rule $\mathcal{L}\forall$ becomes $\forall\mathcal{E}$:

$$\frac{\underline{A}, B[a/\xi] \vdash C}{\underline{A}, \forall\xi.\, B \vdash C}\,\mathcal{L}\forall \qquad \text{is interpreted by} \qquad \begin{array}{c} \underline{A},\ \dfrac{\forall\xi.\, B}{B[a/\xi]}\,\forall\mathcal{E} \\ \vdots \\ C \end{array}$$

5.4 Properties of the translation

The translation from sequent calculus into natural deduction is not 1–1: different proofs give the same deduction, for example

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}\,\mathcal{R}\wedge}{A \wedge A', B \vdash A \wedge B}\,\mathcal{L}1\wedge}{A \wedge A', B \wedge B' \vdash A \wedge B}\,\mathcal{L}1\wedge \qquad \dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}\,\mathcal{R}\wedge}{A, B \wedge B' \vdash A \wedge B}\,\mathcal{L}1\wedge}{A \wedge A', B \wedge B' \vdash A \wedge B}\,\mathcal{L}1\wedge$$

which differ only in the order of the rules, have the same translation:

$$\dfrac{\dfrac{A \wedge A'}{A}\,\wedge 1\mathcal{E} \qquad \dfrac{B \wedge B'}{B}\,\wedge 1\mathcal{E}}{A \wedge B}\,\wedge\mathcal{I}$$

In particular, it would be vain to look for an inverse transformation. What is true is that for a given deduction δ, there is at least one proof in sequent calculus whose translation is δ.

In some sense, we should think of the natural deductions as the true "proof" objects. The sequent calculus is only a system which enable us to work on these objects: $\underline{A} \vdash B$ tells us that we have a deduction of B under the hypotheses $\underline{A}$.

A rule such as the cut

$$\frac{\underline{A} \vdash C \quad \underline{A}', C \vdash B}{\underline{A}, \underline{A}' \vdash B}\,\mathsf{Cut}$$

allows us to construct a new deduction from two others, in a sense made explicit by the translation.

In other words, the system of sequents is not primitive, and the rules of the calculus are in fact more or less complex combinations of rules of natural deduction:

1. The logical rules on the *right* correspond to *introductions*.
2. Those on the *left* to *eliminations*. Here the direction of the rules is inverted in the case of *natural deduction*, since in fact, the tree of natural deduction grows by its leaves at the elimination stage.

   The correspondence $\mathcal{R} = \mathcal{I}$, $\mathcal{L} = \mathcal{E}$ is extremely precise, for example we have $\mathcal{R}\wedge = \wedge\mathcal{I}$ and $\mathcal{L}1\wedge = \wedge 1\mathcal{E}$.
3. The contraction rule $\mathcal{L}\mathsf{C}$ corresponds to the formation of parcels, and $\mathcal{L}\mathsf{W}$, in some cases, to the formation of mock parcels.
4. The exchange rule corresponds to nothing at all.
5. The cut rule does not correspond to a rule of natural deduction, but to the need to make deductions grow at the root. Let us give an example: the strict translation of $\mathcal{L}\Rightarrow$ gives us "from a deduction of A and one of C (with a B-parcel as hypothesis), the deduction

$$\cfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad A \Rightarrow B}{\begin{matrix}B\\ \vdots\\ C\end{matrix}}\,{\Rightarrow}\mathcal{E}$$

   is formed" which grows in the wrong direction (towards the leaves). Yet, the full power of the calculus is only obtained with the "top-down" rule

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A \Rightarrow B\end{matrix}}{B} \Rightarrow\!\mathcal{E}$$

which is the translation of the block of proof:

$$\dfrac{\dfrac{\underline{A}' \vdash A \quad B \vdash B}{\underline{A}', A \Rightarrow B \vdash B}\,\mathcal{L}\!\Rightarrow \qquad \underline{B}' \vdash A \Rightarrow B}{\underline{A}', \underline{B}' \vdash B}\,\mathsf{Cut}$$

The cut corresponds *so* well to a reversal of the direction of the deductions, that, if we translate a cut-free proof, it is almost immediate that the result is a normal deduction. Indeed non-normality comes from a conflict between an introduction and an elimination, which only arises because the two sorts of rules evolve from top to bottom. But just try to produce a redex, writing the introduction rules from top to bottom and the elimination rules from bottom to top! Once again, linear logic clarifies the empirical content of this kind of remark.

We come to the moral equivalence:

$$\text{normal} \;=\; \text{cut-free}$$

In fact, whilst a cut-free proof gives a normal deduction, numerous proofs with cut also give normal deductions, for example

$$\dfrac{A \vdash A \quad A \vdash A}{A \vdash A}\,\mathsf{Cut}$$

is translated by the deduction $\quad A \quad$!

In particular, we see that the sequent calculus sometimes inconveniently complicates situations, by making cuts appear when there is no need. The cut-elimination theorem (Hauptsatz) in fact reiterates the normalisation theorem, but with some technical complications which reflect the lesser purity of the syntax.

As we have already said, every deduction is the translation of some proof, but this proof is not unique. Moreover a normal deduction is the image of a cut-free proof. This is established by induction on the deduction δ of B from parcels of hypotheses $\underline{A}$: we construct a proof π of $\underline{A} \vdash B$ whose translation is δ; moreover, we want π to be cut-free in the case where δ is normal.