5.1 Natural deduction rules for $\land$, $\rightarrow$, $\lor$

There are two rules for each connective. The rules reflect the meanings of the connectives. The easiest is $\land$ (‘and’).

Rules for $\land$

- ($\land$-introduction, or $\land I$) To introduce a formula of the form $A \land B$, you have to have already introduced $A$ and $B$.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$A$</td>
<td>we proved this...</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(other junk)</td>
</tr>
<tr>
<td>2</td>
<td>$B$</td>
<td>and this...</td>
</tr>
<tr>
<td>3</td>
<td>$A \land B$</td>
<td>$\land I(1,2)$</td>
</tr>
</tbody>
</table>

The line numbers are essential for clarity.
Rules for $\land$ ctd.

- ($\land$-elimination, or $\land E$) If you have managed to write down $A \land B$, you can go on to write down $A$ and/or $B$.

1. $A \land B$ we proved this somehow
2. $A$ $\land E(1)$
3. $B$ $\land E(1)$

Rules for $\lor$

- ($\lor$-introduction, or $\lor I$)
To prove $A \lor B$, prove $A$, or (if you prefer) prove $B$.

1. $A$ proved this somehow
2. $A \lor B$ $\lor I(1)$

$B$ can be any formula at all!

1. $B$ proved this somehow
2. $A \lor B$ $\lor I(1)$

$A$ can be any formula at all.
Rules for \( \lor \), ctd.

- (\( \lor \)-elimination, or \( \lor E \)) To prove something from \( A \lor B \), you have to prove it by assuming \( A \), AND prove it by assuming \( B \). (This is arguing by cases.)

\[
\begin{align*}
1 & \quad A \lor B & \quad \text{we got this somehow} \\
2 & \quad \text{ass} & \quad \text{ass} \\
3 & \quad \text{the 1st proof} & \quad \text{the 2nd proof} \\
4 & \quad C & \quad \text{we got it} & \quad \text{we got it again} \\
5 & \quad B & \quad \vdash E (1, 2, 4, 5, 7) \\
8 & \quad C
\end{align*}
\]

The assumptions \( A, B \) are not usable later, so are put in (side-by-side) boxes. *Nothing inside the boxes can be used later.*

Rules for \( \rightarrow \)

- (\( \rightarrow \)-introduction, \( \rightarrow I \): ‘arrow-introduction’) To introduce a formula of the form \( A \rightarrow B \), you assume \( A \) and then prove \( B \).

During the proof, you can use \( A \) as well as anything already established. But *you can’t use \( A \) or anything from the proof of \( B \) from \( A \) later on* (because it was based on an extra assumption).

So we isolate the proof of \( B \) from \( A \), in a box:

\[
\begin{align*}
1 & \quad A & \quad \text{ass} \\
2 & \quad \langle \text{the proof} \rangle & \quad \text{hard struggle} \\
3 & \quad B & \quad \text{we made it!} \\
3 & \quad A \rightarrow B & \quad \vdash I (1, 2)
\end{align*}
\]

*Nothing inside the box can be used later.*

In natural deduction, boxes are used when we make additional assumptions. The first line inside a box should always be labelled ‘ass’ (assumption) — with one exception, coming later (p. 212).
Rules for $\to$, ctd.

- ($\to$-elimination, or $\to E$) If you have managed to write down $A$ and $A \to B$, in either order, you can go on to write down $B$. (This is modus ponens.)

\[
\begin{array}{ccc}
1 & A \to B & \text{we got this somehow...} \\
& & \text{other junk} \\
2 & A & \text{and this too...} \\
3 & B & \to E(1, 2)
\end{array}
\]

5.3 Rules for $\neg$

This is the trickiest case. Also, $\neg$ has three rules! The first two treat $\neg A$ like $A \to \bot$.

- ($\neg$-introduction, $\neg I$) To prove $\neg A$, you assume $A$ and prove $\bot$.

As usual, you can't then use $A$ later on, so enclose the proof of $\bot$ from assumption $A$ in a box:

\[
\begin{array}{ccc}
1 & A & \text{ass} \\
2 & & \text{more hard work, oh no} \\
3 & \bot & \text{we got it!} \\
4 & \neg A & \neg I(1, 3)
\end{array}
\]
Rules for $\lnot$, ctd.

- ($\lnot$-elimination, $\lnot E$)
  From $A$ and $\lnot A$, deduce $\bot$:

1. $\lnot A$ proved this somehow...
2. $\vdash$ junk
3. $A$ ...and this
4. $\bot \lnot E(1,3)$

- ($\lnot \lnot$-elimination, $\lnot \lnot$):
  From $\lnot \lnot A$, deduce $A$. (See example 5.8.)