Natural Deduction for Propositional Logic

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5.1 Natural deduction rules for $\land$, $\rightarrow$, $\lor$

There are two rules for each connective. The rules reflect the meanings of the connectives. The easiest is $\land$ (‘and’).

**Rules for $\land$**

- ($\land$-introduction, or $\land I$) To introduce a formula of the form $A \land B$, you have to have already introduced $A$ and $B$.

1. $A$ we proved this. . .

2. $B$ and this. . .

3. $A \land B$ $\land I(1, 2)$

The line numbers are essential for clarity.
Rules for $\land$ ctd.

- ($\land$-elimination, or $\land E$) If you have managed to write down $A \land B$, you can go on to write down $A$ and/or $B$.

1. $A \land B$ we proved this somehow
2. $A$ $\land E(1)$
3. $B$ $\land E(1)$
Rules for $\lor$

- ($\lor$-introduction, or $\lor I$)
  
  To prove $A \lor B$, prove $A$, or (if you prefer) prove $B$.

  1. $A$ proved this somehow
  2. $A \lor B$ $\lor I(1)$

  $B$ can be any formula at all!

  1. $B$ proved this somehow
  2. $A \lor B$ $\lor I(1)$

  $A$ can be any formula at all.
### Rules for $\lor$, ctd.

- \((\lor\text{-elimination, or } \lor E)\) To prove something from \(A \lor B\), you have to prove it by assuming \(A\), AND prove it by assuming \(B\). (This is arguing by cases.)

<table>
<thead>
<tr>
<th>1</th>
<th>(A \lor B)</th>
<th>we got this somehow</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(A)</td>
<td>ass</td>
</tr>
<tr>
<td>3</td>
<td>(\therefore) the 1st proof</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(C)</td>
<td>we got it</td>
</tr>
<tr>
<td>5</td>
<td>(B)</td>
<td>ass</td>
</tr>
<tr>
<td>6</td>
<td>(\therefore) the 2nd proof</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(C)</td>
<td>we got it again</td>
</tr>
<tr>
<td>8</td>
<td>(C)</td>
<td>(\lor E (1, 2, 4, 5, 7))</td>
</tr>
</tbody>
</table>

The assumptions \(A, B\) are not usable later, so are put in (side-by-side) boxes. **Nothing inside the boxes can be used later.**
Rules for $\rightarrow$

- ($\rightarrow$-introduction, $\rightarrow I$: ‘arrow-introduction’) To introduce a formula of the form $A \rightarrow B$, you assume $A$ and then prove $B$.
During the proof, you can use $A$ as well as anything already established. But you can’t use $A$ or anything from the proof of $B$ from $A$ later on (because it was based on an extra assumption).
So we isolate the proof of $B$ from $A$, in a box:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>ass</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(the proof)</td>
<td>hard struggle</td>
</tr>
<tr>
<td>2</td>
<td>$B$</td>
<td>we made it!</td>
</tr>
<tr>
<td>3</td>
<td>$A \rightarrow B$</td>
<td>$\rightarrow I(1, 2)$</td>
</tr>
</tbody>
</table>

Nothing inside the box can be used later.

In natural deduction, boxes are used when we make additional assumptions. The first line inside a box should always be labelled ‘ass’ (assumption) — with one exception, coming later (p. 212).
Rules for $\rightarrow$, ctd.

- ($\rightarrow$-elimination, or $\rightarrow E$) If you have managed to write down $A$ and $A \rightarrow B$, in either order, you can go on to write down $B$. (This is modus ponens.)

\[
\begin{array}{ll}
1 & A \rightarrow B \quad \text{we got this somehow...} \\
\vdots & \quad \text{other junk} \\
2 & A \quad \text{and this too...} \\
3 & B \quad \rightarrow E(1, 2)
\end{array}
\]
5.3 Rules for $\neg$

This is the trickiest case. Also, $\neg$ has three rules! The first two treat $\neg A$ like $A \rightarrow \bot$.

- ($\neg$-introduction, $\neg I$) To prove $\neg A$, you assume $A$ and prove $\bot$.
  
  As usual, you can't then use $A$ later on, so enclose the proof of $\bot$ from assumption $A$ in a box:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A$</td>
<td>ass</td>
</tr>
<tr>
<td>2</td>
<td>$\bot$</td>
<td>more hard work, oh no</td>
</tr>
<tr>
<td>3</td>
<td>$\bot$</td>
<td>we got it!</td>
</tr>
<tr>
<td>4</td>
<td>$\neg A$</td>
<td>$\neg I(1, 3)$</td>
</tr>
</tbody>
</table>
Rules for $\neg$, ctd.

- ($\neg$-elimination, $\neg E$)
  From $A$ and $\neg A$, deduce $\bot$:

  1 $\neg A$ proved this somehow…
  2 : junk
  3 $A$ …and this
  4 $\bot$ $\neg E(1, 3)$

- ($\neg\neg$-elimination, $\neg\neg$):
  From $\neg\neg A$, deduce $A$. (See example 5.8.)