CS112 – Tuesday, September 8, 2008

- Agenda
  - Reminders
  - Introduction to First Order Logic
  - Natural Deduction for FOL
- Reminders
  - Problem Set 1 on website; due September 23
    - Group Policy: Your work on this assignment should be your own!
  - Office Hours
    - James' office hours are cancelled this week because he is away at a workshop, but Jess' are as scheduled: Wednesday and Thursday 2pm to 3:30pm.
    - If you plan on attending, please try to let me know in advance.
    - If those times don't work for you, feel free to e-mail your questions or set up an appointment.
- Introduction to FOL
  - Simple Translations
    - Dx: x is a dog
    - Yx: x is yellow
    - Bx: x is black
    - h: Homer
    - o: Otis
    - Homer is a yellow dog. → Dh & Yh
    - If Homer is yellow, then Otis is black. → Yh -> Bo
  - Introducing Quantifiers
    - ∀ -- universal quantifier
    - All dogs are yellow. → ∀x(Dx -> Yx)
    - ∃ -- existential quantifier
    - Some dogs are yellow → ∃x(Dx & Yx)
  - What can FOL do that PL can't do?

    - Much more expressive; gets us closer to being able to model real language; we'll see next week that it has the same nice properties that PL has (soundness and completeness)
  - What can't FOL do?
    - Not expressive enough, as we'll see on Friday
- On Friday, we'll do translations into and out of FOL as well as some basic FOL semantics.
  - Today, we're going to introduce the last 4 rules of natural deduction you need to do ND proofs in FOL, while the other rules are fresh in your mind.
- Quantifier Equivalences
  - Laws of quantifier negation (negations are scary!)
  - Laws of quantifier independence (note the last one only goes in one direction!)
  - Laws of quantifier movement
    - What does "x is not free in φ" mean? (Scoping)
- New ND Rules
  - Each quantifier gets an introduction and an elimination rule
  - The Easy Ones:
    - Universal Elimination (∀e) – Eliminate the universal quantifier and replace all of the variables it bound with a constant.

      $$\begin{array}{|l} (\forall x)P \\ \hline P(a/x) \end{array}$$

      - (∃y)Hay is a substitution instance of (∀x)(∃y)Hxy
      - Hab is not a substitution instance of (∀x)(∃y)Hxy because only one substitution can be performed at once and it can only be performed on the initial quantifier
      - ∃xHxb is not a substitution instance of ∃x∀yHxy because you can only use ∀e on a universal sentence. This one is an existential sentence. (You could use an equivalence rule here, but that's not what I'm trying to show here.)

- ∀e can be used on any universal sentence (the main connective has to be a universal).
- Make sure you replace every newly free variable with a constant
- Example:

1	$(\forall y)(Py \supset Sy)$	P
2	Ps	P ⊢ Ss
3	$Ps \supset Ss$	1 ∀E
4	Ss	1,3 ⊃E

▪ Existential Introduction (∃i) – Replace a substitution instance (or just a constant in a sentence) with an existential quantifier.

$P(a/x)$

$(\exists x)P$

- ∃i does not require that every occurrence of an individual constant be existentially generalized
- Remember that 'Rmm' can be a substitution instance of 3 different sentences: (∃x)Rxx, (∃x)Rxm, and (∃x)Rmx
  - ♦ Any one of these three sentences could be derived from 'Rmm' using ∃I
- Example:

1	$Mg\&Pg$	P
2	$(\exists x)(Mx\&Px)$	1 ∃I

- Example 2: (∀x)(∀y)Cxy |-- (∃x)(∃y)Cxy

➢ The Hard Ones:

▪ Universal Introduction (∀i) – Prove that a sentence is true for every possible constant by proving it for some arbitrary constant

- What makes a constant arbitrary?
  - ♦ The constant cannot appear in any sentence that the derivation depends on. (Can't be in the goal sentence or any undischarged assumptions)

$P(a/x)$

$(\forall x)P$

- Provided that:
  - **a** does not occur in an undischarged assumption
  - **a** does not occur in (∀**x**)**P**

- Example:

1	$(\forall x)(Dx \supset Ox)$	P
2	$(\forall x)(Ox \supset Sx)$	P ⊢ (∀x)(Dx ⊃ Sx)
3	$\quad Da$	PA (for ⊃I) ⊢ Sa
4	$\quad Da \supset Oa$	1 ∀E
5	$\quad Oa$	3,4 ⊃E
6	$\quad Oa \supset Sa$	2 ∀E
7	$\quad Sa$	5,6 ⊃E
8	$Da \supset Sa$	3-7 ⊃I
9	$(\forall x)(Dx \supset Sx)$	9 ∀I

- Why is ∀i hard?
  - This one is hard because it takes a big leap of faith. It's a pretty controversial rule, but it's generally been accepted. If you're interested in learning about the controversy, let me know and we can dig into it, but for now, these rules are just for pushing symbols around, so just think of this rule as a way to prove universal sentences.

- Existential Elimination (∃e) – Prove your desired conclusion by assuming an arbitrary instance of an existential sentence in your proof.
  - This rule is used just like we used disjunction elimination (proof by cases). We use it to exploit (eliminate) an existential, but that sentence doesn't really have to have anything to do with the thing we're trying to prove.
  - Use this rule whenever an existential sentence is in the premises.

$(\exists x)P$

$\quad P(a/x)$

$\quad Q$

Q

- Provided that:
  - **a** does not occur in an undischarged assumption

- **a** does not occur in (∃**x**)**P**
- **a** does not occur in **Q**

- Example:

1	$(\forall x)(Gx \supset Ge)$	P
2	$(\exists x)Gx$	P ⊢ Ge
3	Ga	PA (for ∃E)
4	$Ga \supset Ge$	1 ∀E
5	Ge	3,4 ⊃E
6	Ge	2, 3-5 ∃E

- Big Example: Cf & Bfl, (∃x)(Cx & Bxf), (∀x)(∀y)(∀z)[(Bxy & Byz) ⊃ Bxz] |-- (∃z)[Cz & (Bzf & Bzl)]