Spatial Reasoning with a Hole

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Abstract. Cavities in spatial phenomena require geometric representations of regions with holes. Existing models for reasoning over topological relations either exclude such specialized regions (9-intersection) or treat them indistinguishably from regions without holes (RCC-8). This paper highlights that inferences over a region with a hole need to be made separately from, and in addition to, the inferences over regions without holes. First the set of 23 topological relations between a region and a region with a hole is derived systematically. Then these relations' compositions over the region with the hole are calculated so that the inferences can be compared with the compositions of the topological relations over regions without holes. For 266 out of the 529 compositions the results over the region with the hole were more detailed than the corresponding results over regions without holes, with 95 of these refined cases providing even a unique result. In 27 cases, this refinement up to uniqueness compares with a completely undetermined inference for the relations over regions without holes.

1 Introduction

Some spatial phenomena have cavities (Figure 1a-d), which require geometric representations of regions with holes when they are modeled in geographic information systems. Most prominent geographic examples are the territorial configurations of South Africa (which completely surrounds Lesotho), the former East Germany (completely surrounding West Berlin), and Italy (which completely surrounds San Marino and the Vatican City). Many other spatial configurations with holes have been thought of [1]. This paper focuses on the topological relations involving regions with a single hole.

Although such regions with holes resemble visually regions with indeterminate (i.e., broad) boundaries (Figure 2a), their topologies differ conceptually, since a region with a broad boundary is an open set (Figure 2b), whereas a region with a hole is a closed set (Figure 2c). Therefore, the various versions of topological relations between regions with broad boundaries [2,3,13] do not apply immediately to regions with holes.

While geometric models of spatial features have matured to capture appropriately the semantics of regions with holes [11,17], models of qualitative spatial relations over regions with holes have essentially stayed in their infancy. Some models of

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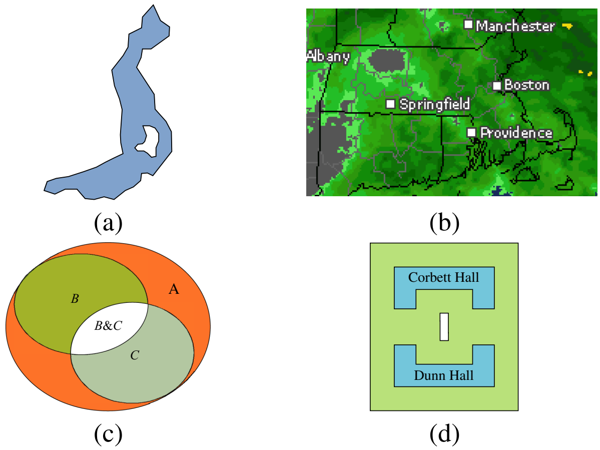
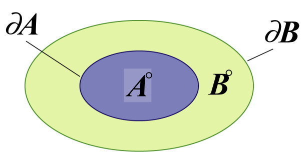


Fig. 1. Regions with a hole: (a) the Lago Iseo (with Monte Isola, Italy's largest island in a lake), (b) the area of Massachusetts with precipitation on December 23, 2006 at 7:30am, (c) the part of region A that cannot be reached by both persons *B* and *C* when they were to travel from their current locations for a set amount of time, and (d) the smoke-free zone around Corbett Hall and Dunn Hall on the UMaine campus implied by a 30ft buffer zone around each building



(a)

Broad-Boundary Region

$$\partial \boldsymbol{R}_{bb} = \boldsymbol{B}^\circ \setminus \boldsymbol{A}^\circ$$
$$\boldsymbol{R}_{bb}^\circ = \boldsymbol{A}^\circ$$
$$\boldsymbol{R}_{bb}^- = (\boldsymbol{B}^- \cup \partial \boldsymbol{B})$$

(b)

Region with Hole

$$\partial \boldsymbol{R}_h = \partial \boldsymbol{B} \cup \partial \boldsymbol{A}$$
$$\boldsymbol{R}_h^\circ = \boldsymbol{B}^\circ \setminus (\boldsymbol{A}^\circ \cup \partial \boldsymbol{A}^\circ)$$
$$\boldsymbol{R}_h^- = \boldsymbol{B}^- \cup \boldsymbol{A}^\circ$$

(c)

Fig. 2. Two closed discs (*A* and *B*): (a) their topological components boundary (∂R), interior (R°), and exterior (R^-) that contribute to the formation of (b) a region with a broad boundary (R_{bb}) and (c) a region with a hole (R_h)

topological relations have addressed regions with holes [8,14,20], distinguishing varying levels of details about the placements of the holes, however, qualitative spatial reasoning with such relations has been either discarded or treated like reasoning without holes. The 4-intersection [9] and 9-intersection [10], for instance, exclude explicitly as the relations' domain and co-domain any regions with holes, so that the comprehensive body of inferences over topological relations based on the 9-intersection does not apply directly to regions with holes. On the other hand, the region-connection calculus (RCC) [19] makes no explicit distinction between regions with or without holes so that this model of topological relations, and their inferences from compositions, applies to regions with holes as well. While the 9-intersection composition table is extentional [15], RCC's applicability to regions with or without holes has given rise to a non-extentional composition table. These differences in the composition of topological relations are the key justification for the need to differentiate relations for regions with holes from relations for regions without holes when making inferences over topological relations.

The following example highlights the need for such an explicit distinction. Given a region B such that it *overlaps* with a region A and also *overlaps* with a region C. From the composition A *overlaps* B and B *overlaps* C [6] one can deduce the possible relations between A and C (Figure 3a-h), yielding in this case the universal relation U_8 (i.e., all eight topological relations are possible).

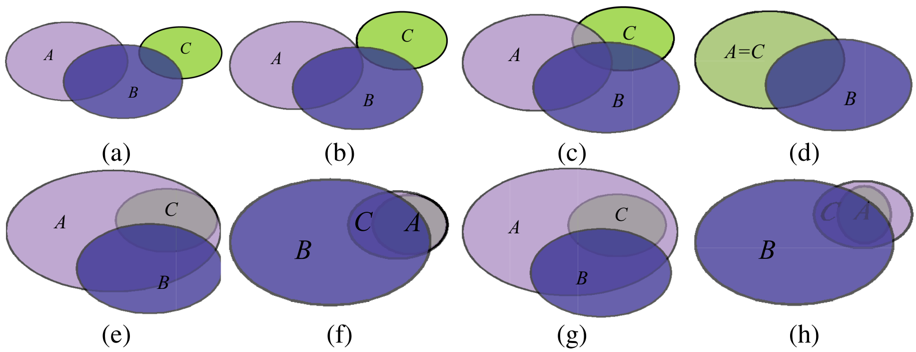
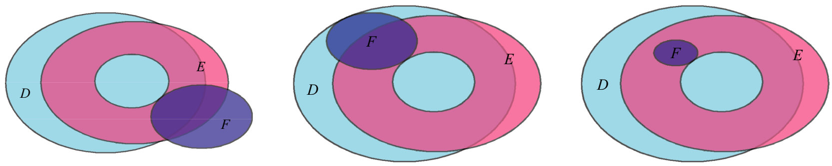


Fig. 3. The eight possible configurations if region A *overlaps* region B and B *overlaps* region C: (a) A *disjoint* C, (b) A *meet* C, (c) A *overlaps* C, (d) A *equal* C, (e) A *covers* C, (f) A *coveredBy* C, (g) A *contains* C, and (h) A *inside* C

If one starts, however, with a region with a hole (E) that overlaps with two other regions, D and F, each without holes, the hole may play a significant role in constraining the possible relations between D and F. For instance, let E *overlap* with D such that E's hole is completely contained in D, and let E overlap with F such that F meets E's hole. Then D could *overlap* with F (Figure 4a), D could *cover* F (Figure 4b), and D could *contain* F (Figure 4c). Therefore, the insertion of a hole into the first region results in a composition scenario that is more constrained than the one without the hole. Treating both cases with the same (less constrained) composition would offer some incorrect choices in case the region has a hole.

Is this example an anomaly? Or maybe even the only case in which relation inferences differ for regions and regions with a hole? Or are there so many more cases that typically the reasoning over a region with a hole differs from the well known topological inferences of regions without holes? This paper provides answers to these questions through a systematic study of the topological relations involving a region with a hole, the derivation of the composition inferences of these relations, and a quantitative comparison of these compositions with the compositions of topological relations between regions without holes [6]. The topological relation between a region and a region with a hole is denoted by t_{RR_h} (and its converse relation by t_{R_hR}), while t_{RR} refers to the topological relation between two regions (each without a hole).



(a) (b) (c)

Fig. 4. The three possible configurations if E *overlaps* with D such that E's hole is *inside* D, and E also overlaps with F such that E's hole *meets* F: (a) D *overlaps* F, (b) D *covers* F, and (c) D *contains* F

B° is B's interior
B^{-1} is the inner exterior of B, which fills B's hole
B^{-0} is the outer exterior of B
$\partial_1 B$ is the inner boundary of B, which separates B° from B^{-1}
$\partial_0 B$ is the outer boundary of B, which separates B° from B^{-0}

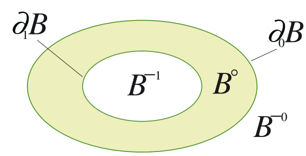


Fig. 5. B's five topologically distinct and mutually exclusive parts

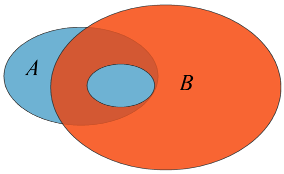
Throughout this paper, the qualitative model of a region with a hole (B) is based on B's five topologically distinct and mutually exclusive parts (Figure 5).

The elements of the qualitative description of a region with a hole are (1) its hole B_H ($B^{-1} \cup \partial_1 B$) and (2) the generalized region B^* ($B_H \cup B^\circ \cup \partial_0 B$). B^* and B_H are spatial regions, that is, each region is homeomorphic to a 2-disk so that the eight t_{RR} [9] apply to B^* and B_H (but not to B, because B with the hole is not homeomorphic to a 2-disk). The topological relation between B^* and B_H is *contains*, therefore, this is a more restrictive model than the generic region-with-holes model [8], where B_H also could have been *coveredBy* or even *equal* to B^*, thereby leading to somewhat different semantics of a region with a hole.

The remainder of this paper is organized as follows: Section 2 specifies the canonical model used for modeling a region with a hole as well as such a region's topological relation with another region. Section 3 presents a method to derive the t_{RR_h} that are feasible between a region and a region with a hole. Section 4 presents the 23 relations that can be found between a region and a region with a hole, followed by an analysis of these relations' algebraic properties in Section 5. Section 6 derives the qualitative inferences that can be made with t_{RR_h} and t_{R_hR}, focusing on compositions over a common region with a hole. Section 7 analyzes these compositions, comparing their reasoning power with the compositions of topological relations between regions without holes. The paper closes with conclusions and a discussion of future work in Section 8.

2 Qualitative Model of a Region with a Hole

The topological relation between a region (A) and a region with a hole (B) is modeled as a *spatial scene* [5], comprising A, B^*, and B_H together with nine binary topological relations among these three regions (Figure 6).



(a)

	A	B^*	B_H
A	*equal*	*overlaps*	*covers*
B^*	*overlaps*	*equal*	*contains*
B_H	*coveredBy*	*inside*	*equal*

(b)

Fig. 6. Topological relation of a region with a hole: (a) graphical depiction of a configuration and (b) the corresponding symbolic description as a spatial scene

In such a spatial scene, five of the nine binary topological relations are implied for any configuration between a region and a region with a hole: each region is *equal* to itself, B^* *contains* B_H, conversely B_H is *inside* B^*, and for the two relations between A and B^* and A and B_H, their converse relations (from B^* to A and from B_H to A) are implied by the arc consistency constraint [16]; therefore, a model of such a spatial scene only requires the explicit specification of the two relations between A and B^* and A and B_H to denote t_{RR_h} (Eqn. 1). These relations between A and B^* and A and B_H are called the *constituent relations* of a topological relation between a region and a region with a hole. Their horizontal 1x2 matrix is a direct projection of the top elements in the two right-most columns of the spatial scene description.

$$t_{RR_h}(A, B) = \left[t(A, B^*) \quad t(A, B_H) \right] \tag{1}$$

The principal relation $\pi(t_{RR_h})$ is then the first element of t_{RR_h} (Eqn. 2).

$$\pi(t_{RR_h}(A, B)) = r(A, B^*) \tag{2}$$

Section 3.3 shows that some configurations actually only require the principal relation in order to specify t_{RR_h} completely.

3 Deriving the Topological Relations Between a Region and a Region with a Hole

The spatial scene can also be used for the derivation of what topological relations actually exist between a region and a region with a hole. Since two of the scene's nine topological relations are subject to variations (the relations between A and B^* and A and B_H), a total of 8^2=64 t_{RR_h} could be specified. But only a subset of these 64 relations is feasible. For example, t_{RR_h} $[$*contains* *disjoint*$]$ is infeasible, because B^*

cannot be *inside* A at the same time as B_H (which is inside B^*) is *disjoint* from A. Therefore, a topological relation from a region with a hole to another region is feasible if (1) that scene's representation is consistent and (2) there exists a corresponding graphical depiction.

The binary topological relation between a region (A) and a region with a hole (B) is established as a 3-region scene comprising A, B^*, and B_H with the constraint that B^* *contains* B_H (Figure 7). The topological relation between a region and a region with a hole holds if this 3-region scene is node-consistent, arc-consistent, and path-consistent (Macworth 1977) for the four values t (A, B^*), t (A, B_H) and their corresponding converse relations t (B^*, A) and t (B_H, A).

	A	B^*	B_H
A	*equal*	$t\ (A, B^*)$	$t\ (A, B_H)$
B^*	$t\ (B^*, A)$	*equal*	*contains*
B_H	$t\ (B_H, A)$	*inside*	*equal*

Fig. 7. A 3-region spatial scene that captures the constituent relations of a binary topological relation between a region (A) and a region with a hole (B)

The range of these four relations is the set of the eight t_{RR}. With four variables over this domain, a total of $8^4 = 4{,}096$ configurations could be described for the topological relations between a region and a region with a hole. Only a subset of them is feasible, however. These feasible configurations are those whose 3-region scenes are consistent. Since in the feasible configurations t (A, B^*) must be equal to the converse of t (B^*, A), the enumeration of the relations in the feasible configuration can be reduced. The same converseness constraint also holds for t (A, B_H) and t (B_H, A); therefore, for a feasible t_{RR_h} two of the four relations are implied. Thus, only two of the four unknown relations are necessary to completely describe a feasible t_{RR_h}, reducing the number of possible configurations to $8^2 = 64$.

4 Twenty-Three Relations Between a Region and a Region with a Hole

In order to determine systematically the feasible t_{RR_h}, a scene consistency checker has been implemented, which iterates for each unknown (i.e., universal) relation over the eight possible relations and determines whether that spatial scene is node-consistent, arc-consistent, and path-consistent [16]. Only those configurations that fulfill all three consistency constraints are candidates for a valid t_{RR_h}. Twenty-three spatial scenes representing a region and a region with a hole have been found to be consistent (Figure 8).

The remaining 64–23=41 candidate configurations for t_{RR_h} have been found to be inconsistent. Therefore, the 23 consistent cases establish the 23 binary topological relations between a region and a region with a hole.

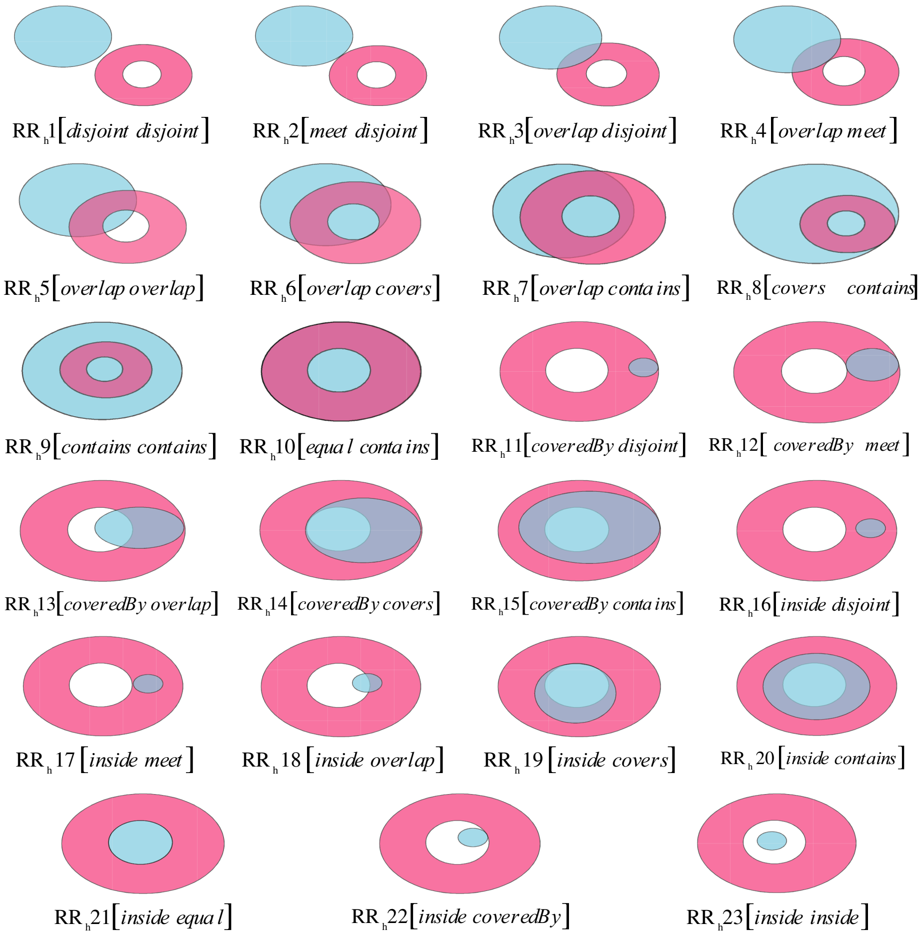


Fig. 8. Graphical depictions of the 23 topological relations between a region and a region with a hole

5 Properties of the Twenty-Three Relations

These 23 t_{RR_h} can be viewed as refinements of the eight t_{RR}. Five of the eight t_{RR}—*disjoint*, *meet*, *covers*, *contains*, *equal*—do not reveal further details if region B has a hole, because in each of these cases the relation between A and B^* is so strongly constrained that only a single relation is possible between A and B's hole B_H. The remaining three t_{RR}—*overlap*, *coveredBy*, *inside*—are less constraining as each offers multiple variations for the topological relations between A and B_H: *overlap* and *coveredBy* each have five variations for $t\,(A,\,B_H)$, while *inside* has a total of eight variations.

Without a specification of the relation between *A* and B_H, the configurations *A overlap B** (RR_h5– RR_h8) and *A coveredBy B** (RR_h11–RR_h15) are underdetermined, that is, one can only exclude for each case the three relations *A equal* B_H, *A coveredBy* B_H, and *A inside* B_H, but cannot pin down which of the remaining five choices—*A disjoint* B_H, *A meet* B_H, *A overlap* B_H, *A covers* B_H, *A contains* B_H—actually holds. Likewise, the configuration *A inside B** is undetermined without a specification of the relation between *A* and B_H, because any of the eight t_{RR} could hold between *A* and B_H.

5.1 Converse Relations

Since the domain and co-domain of t_{RR_h} refer to different types—a region with a hole and a region without a hole—there is neither an identity relation, nor are there symmetric, reflexive, or transitive t_{RR_h}. The concept of a converse relation (i.e., the relation between a region with a hole and another region) still exists, however. The relation converse to t_{RR_h} is implied through the converse property of the constituent relations (Eqn. 1)—$t(B^*,A)=\overline{t(A,B^*)}$ and $t(B_H,A)=\overline{t(A,B_H)}$—which is captured in a transposed matrix of the constituent relations (Eqn. 3).

$$t_{R_hR}(B,A)=\begin{bmatrix} t(B^*,A) \\ t(B_H,A) \end{bmatrix}=\begin{bmatrix}\overline{t(A,B^*)} & \overline{t(A,B_H)}\end{bmatrix}^T \tag{3}$$

This leads immediately to 23 t_{R_hR}. Their names are chosen systematically so that all pairs of converse relations have the same index (Eqn. 4).

$$\forall x:1\ldots 23:\quad \mathrm{R}_h\mathrm{R}x=\overline{\mathrm{RR}_h x} \tag{4}$$

From among the 23 pairs of converse t_{RR_h} t_{R_hR}, five relation pairs have identical constituent relations (Equations 5a-e), because each element of these five pairs has a symmetric converse relation, that is, $\mathrm{R}_h\mathrm{R}x=\left(\mathrm{RR}_h x\right)^T$.

$$\begin{bmatrix} disjoint \\ disjoint \end{bmatrix}^T=\begin{bmatrix} disjoint & disjoint \end{bmatrix} \tag{5a}$$

$$\begin{bmatrix} meet \\ disjoint \end{bmatrix}^T=\begin{bmatrix} meet & disjoint \end{bmatrix} \tag{5b}$$

$$\begin{bmatrix} overlap \\ disjoint \end{bmatrix}^T=\begin{bmatrix} overlap & disjoint \end{bmatrix} \tag{5c}$$

$$\begin{bmatrix} overlap \\ meet \end{bmatrix}^T=\begin{bmatrix} overlap & meet \end{bmatrix} \tag{5d}$$

$$\begin{bmatrix} overlap \\ overlap \end{bmatrix}^T=\begin{bmatrix} overlap & overlap \end{bmatrix} \tag{5e}$$

5.2 Implied Relations

The dependencies among a region's relations to the generalized region and the hole reveal various levels of constrains (Figure 9). While five $t\,(A, B^*)$ imply a unique relation for $t\,(A, B_H)$, two other $t\,(A, B^*)$ restrict $t\,(A, B_H)$ to five choices. Only one $t\,(A, B^*)$—*inside*—yields the universal relation U_8, imposing no constraints on $t\,(A, B_H)$.

Known Relation $t\,(A, B^*)$	Implied Relation $t\,(A, B_H)$
disjoint	*disjoint*
meet	*disjoint*
covers	*contains*
contains	*contains*
equal	*contains*
overlap	not {*equal*, *coveredBy*, *inside*}
coveredBy	not {*equal*, *coveredBy*, *inside*}
inside	U

Fig. 9. Constraints imposed by a specified $t\,(A, B^*)$ on $t\,(A, B_H)$

Reversely, knowledge of the relation $t\,(A, B_H)$ implies in three cases—if $t\,(A, B_H)$ is *equal*, *coveredBy*, or *inside*—a unique relation between A and B^*; has three choices for three relations between A and B^* (if $t\,(A, B_H)$ is *meet*, *overlap*, or *covers*); five choices in one case (if $t\,(A, B_H)$ is *disjoint*); and six choices if $t\,(A, B_H)$ is *contains* (Figure 10).

Known Relation $t\,(A, B_H)$	Implied Relation $t\,(A, B^*)$
equal	*inside*
coveredBy	*inside*
inside	*inside*
disjoint	not {*equal*, *covers*, *contains*}
meet	{*overlap*, *coveredBy*, *inside*}
overlap	{*overlap*, *coveredBy*, *inside*}
covers	{*overlap*, *coveredBy*, *inside*}
contains	not {*disjoint*, *meet*}

Fig. 10. Constraints imposed by a specified $t\,(A, B_H)$ on $t\,(A, B^*)$

The dependencies may be seen as an opportunity for minimizing the number of relations that are recorded. For example, if one of the two implications were such that all known relations implied a unique relation, then it would be sufficient to record only the known relation, thereby cutting into half the amount of relations to be stored for each t_{RR_h}. Such a simple choice does not apply, however. Since five $t\,(A, B_H)$ are implied uniquely by $t\,(A, B^*)$, $t\,(A, B_H)$ needs to be recorded only in three cases to fix a complete t_{RR_h} specification. Reversely only three $t\,(A, B^*)$ are implied uniquely by their $t(A, B_H)$. Therefore, the common-sense choice of favoring the relation with respect to the generalized region over the relation to the hole gets further support.

6 Compositions over a Region with a Hole

A key inference mechanism for relations is their composition, that is, the derivation of the relation A to C from the knowledge of the two *relations* $t(A, B)$ and $t(B, C)$. A complete account of all relevant compositions considers first all combinatorial compositions of relations with regions (R) and regions with a hole (R_H). Since all compositions involve two binary relations (i.e., _ _ ; _ _), each over a pair of R and R_H, there are $2^4 = 16$ possible combinations (Figure 11). Eight of these sixteen combinations specify invalid compositions (C 3–6 and C 11–14), because the domain and co-domain of the composing relations' common argument are of different types (i.e., trying to form a composition over a Region and a Region with a hole). Among the remaining eight combinations, C 1 is the well-known composition of region-region relations. Two pairs of combinations capture converse compositions—C 2 and C 9, as well as C 8 and C 15—while three combinations capture symmetric compositions—C 7, C 10, and C 16.

C 1	$t_{RR};\ t_{RR}$	C 5	—	C 9	$t_{R_hR};t_{RR}$	C 13	—
C 2	$t_{RR};\ t_{RR_h}$	C 6	—	C 10	$t_{R_hR};t_{RR_h}$	C 14	—
C 3	—	C 7	$t_{RR_h};t_{R_hR}$	C 11	—	C 15	$t_{R_hR_h};t_{R_hR}$
C 4	—	C 8	$t_{RR_h};t_{R_hR_h}$	C 12	—	C 16	$t_{R_hR_h};t_{R_hR_h}$

Fig. 11. The 16 combinations of compositions of binary relations with regions (R) and regions with a hole (R_H)

From among these combinations of compositions involving a region with a hole, we focus here on Comp 7, the inferences from t_{RR_h} ; t_{R_hR}. A spatial scene serves again as the framework for a computational derivation of all compositions. Objects A and C are two regions without a hole, whereas object B is a region with a hole. The corresponding spatial scene has four regions (A, B^*, B_H, and C) with their sixteen region-region relations (Figure 12). The pair of relations $t(A, B^*)$ $t(A, B_H)$ must be a subset of the 23 valid t_{RR_h}, while the pair of relations $t(B^*, C)$ $t(B_H, C)$ must be a subset of the 23 valid t_{R_hR}. Furthermore, $t(B^*, A)$ and $t(B_H, \mathrm{A})$ must be the respective converse relations of $t(A, B^*)$ and $t(A, B_H)$. The same converse property must hold for the pair $t(C, B^*)$ $t(C, B_H)$ with respect to $t(B^*, C)$ $t(B_H, C)$. With 23 pairs for each t_{RR_h} and t_{R_hR}, there are 529 compositions. The range of the inferred relation $t(A, C)$ is the set of the eight t_{RR}. This composition of $t(A, B)$; $t(B, C)$ is specified for any spatial scene that is node-consistent, arc-consistent, and path-consistent. To determine systematically all consistent compositions, we have developed a software prototype of a consistency checker that evaluates a spatial scene

	A	B^*	B_H	C
A	*equal*	U	U	U
B^*	U	*equal*	*contains*	U
B_H	U	*inside*	*equal*	U
C	U	U	U	*equal*

Fig. 12. The spatial scene over four regions used for the derivation of the composition $t\,(A, B)$; $t\,(B, C)$

for the three consistencies. All compositions where found to be valid (i.e., none of the compositions resulted in the empty relation).

Figures 14a and 14b summarize the result graphically, using for the composition the iconic representation of the region-region relations based on their conceptual neighborhood graph [12]. A highlighted relation in that graph indicates that that relation is part of the particular composition. The universal relation U_8 is then an icon with all relations highlighted (Figure 13a), while a unique inference has a single relation highlighted (Figure 13b). The composing relations t_{RR_h} and t_{R_hR} are also captured by the same neighborhood graph, in which the relation to the generalized region is superimposed over the relation to the hole (Figure 13c).

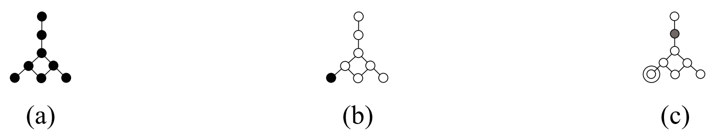


Fig. 13. Iconic representation of relations and compositions: (a) universal relation of region-region relations, (b) unique composition result (*inside*) of region-region relations, and (c) unique t_{RR_h} with the large circle identifying the relation between region A and the generalized region B^* and the black disc highlighting the relation between A and B_H

7 Analysis of Compositions

The 64 compositions of t_{RR} ; t_{RR} [6] form the benchmark for the assessment of the reasoning power of compositions involving regions with holes.

Finding 1: The composition table t_{RR_h} ; t_{R_hR} (Figure 14a and 14b) shows that all 529 compositions are valid (i.e., there is no empty relation as the result of any of the compositions). This means none of the 529 4-object scenes considered to calculate the compositions (Figure 12) is inconsistent. The same level of consistency was also found for the t_{RR} ; t_{RR} composition table.

Finding 2: All compositions are compatible with the composition results of their principal relations (Eqn. 6), that is, the inferences from the principal relations provide an upper bound for the reasoning over regions with a hole.

$$\forall a : 1 \ldots 23, \forall b : 1 \ldots 23 : \quad \mathrm{RR}_h a \,;\, \mathrm{R}_h \mathrm{R} b \subseteq \pi(\mathrm{RR}_h a) \,;\, \pi(\mathrm{R}_h \mathrm{R} b) \qquad (6)$$

Finding 3: Among the 529 compositions there are 263 (49.7%) whose results are identical to the compositions of the relations' principal relations (Eqn. 7). Therefore, for slightly less than half of the inferences the hole is immaterial, while it matters for the remaining 266 inferences.

$$\exists a,b \mid a \neq b: \quad \mathrm{RR}_h a \,;\, \mathrm{R}_h \mathrm{R} b = \pi(\mathrm{RR}_h a) \,;\, \pi(\mathrm{R}_h \mathrm{R} b) \tag{7}$$

Finding 4: Among the 266 compositions whose results are more refined than the compositions of their principal relations, 95 compositions are refined to uniqueness (Eqn. 8). If one were to resort in these cases to the compositions of their principal relations, one would incorrectly infer that these compositions are underdetermined.

$$\exists a,b \mid a \neq b: \mathrm{RR}_h a \,;\, \mathrm{R}_h \mathrm{R} b \subset \pi(\mathrm{RR}_h a) \,;\, \pi(\mathrm{R}_h \mathrm{R} b) \wedge \#(\mathrm{RR}_h a \,;\, \mathrm{R}_h \mathrm{R} b) = 1 \tag{8}$$

To further assess the inference power of the compositions, we use the composition's *cardinality* (Eqn. 9a), which is the count of relations in that composition result, and the *composition table's cardinality* (Eqn. 9b), which is the sum of the cardinalities of all compositions in a table. This yields the *composition table's normalized crispness* (Eqn. 9c), whose lowest value of 0 stands for compositions that result in the universal relation and whose value increases linearly for composition results with fewer choices. The latter measure also applies to subsets of a composition table to assess and compare the inferences of particular groups of relations. The corresponding measures for $t_{RR} \,;\, t_{RR}$ can be defined accordingly.

$$card_{23}^{ij} = \#(\mathrm{RRh}i;\mathrm{RhR}j) \tag{9a}$$

$$\gamma_{23} = \sum_{i=1..\#(U_{23})}^{j=1..\#(U_{23})} card_{23}^{ij} \tag{9b}$$

$$\overline{\Gamma_{23}} = 1 - \frac{\gamma_{23}}{\#(U_8) * \#(U_{23}) * \#(U_{23})} \tag{9c}$$

Finding 5: While the cardinality of the $t_{RR_h} \,;\, t_{R_hR}$ composition table is over seven times higher than that of the $t_{RR};t_{RR}$ composition table (γ_{23} =1389 vs. γ_8 =193), the overall inferences from $t_{RR_h} \,;\, t_{R_hR}$ are crisper, because the average composition cardinality is approximately 8% higher for all $t_{RR_h} \,;\, t_{R_hR}$ than for all $t_{RR};t_{RR}$ ($\overline{\Gamma_{23}}$ =0.67 vs. $\overline{\Gamma_8}$ =0.62).

Finding 6: The increase in crispness is primarily due to a decrease in the relative number of compositions with a cardinality of 5 (and to a lesser degree cardinalities 6 and 8), while simultaneously the relative numbers of compositions with cardinalities 3, 2, and 4 (and to a miniscule amount those of compositions with cardinality 1) increase (Figure 15). Overall 239 ambiguities of pure topological reasoning are reduced, but not fully eliminated, when considering the holes in the regions.

Finding 7: In absolute numbers the count of compositions with unique results goes up from 27 in $t_{RR} \,;\, t_{RR}$ to 224 in $t_{RR_h} \,;\, t_{R_hR}$. Since—for a different set of relations, though—people have been found to make composition inferences more correctly if the result is unique [19], this increase augurs well for people's performance when reasoning over relations with holes.

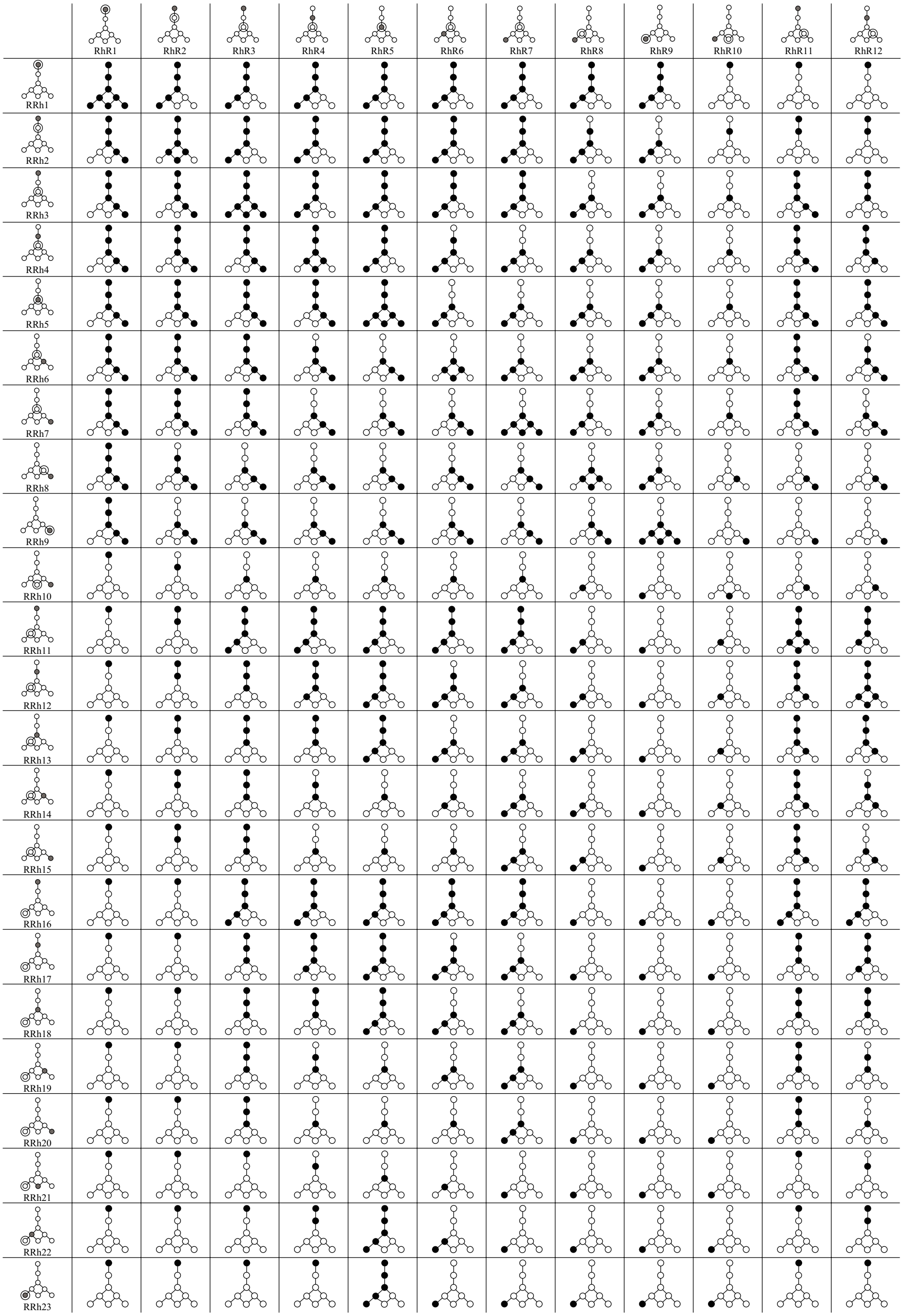


Fig. 14a. Composition table t_{RR_h} ; t_{R_hR} (for $t_{R_hR} = [R_hR1 \ldots R_hR12]$)

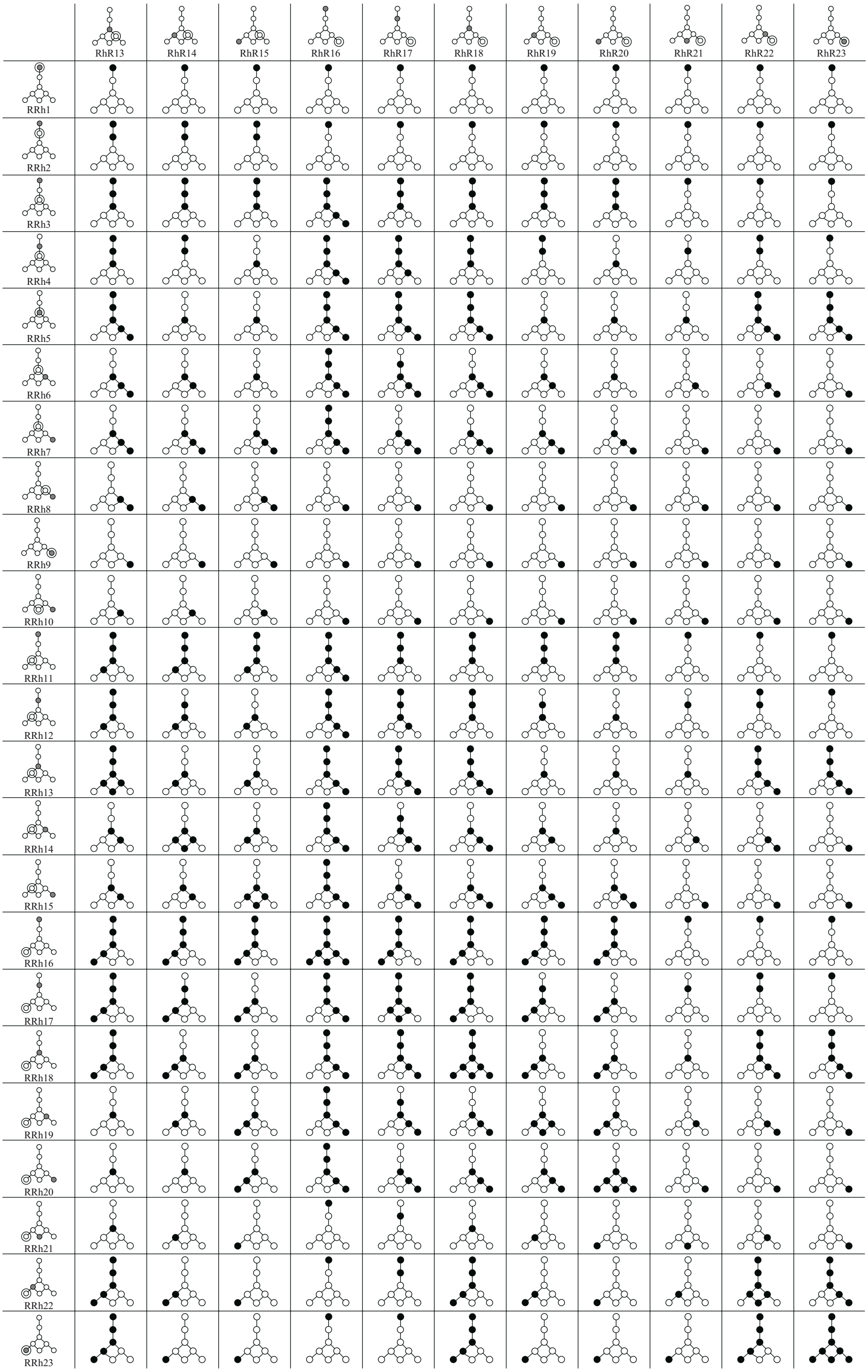


Fig. 14b. Composition table t_{RR_h} ; t_{R_hR} (for $t_{R_hR} = [R_hR13 \ldots R_hR23]$)

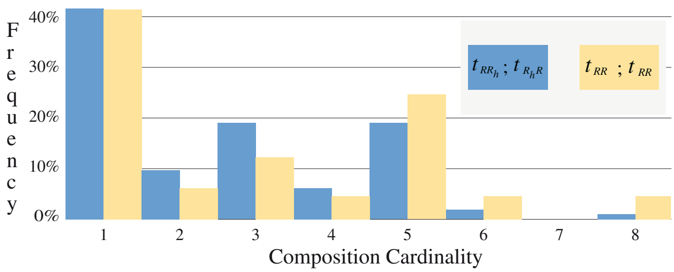


Fig. 15. Comparison of the frequencies of compositions results with cardinality 1 (unique inference) through 8 (universal relation) for composition table $t_{RR_h} ; t_{R_hR}$ with t_{RR} ; t_{RR}

		RhR1 *d d*	RhR2 *m d*	RhR3 *o d*	RhR4 *o m*	RhR5 *o o*	RhR6 *o cB*	RhR7 *o i*	RhR8 *cB i*	RhR9 *i i*	RhR10 *e i*	RhR11 *cv d*	RhR12 *cv m*	RhR13 *cv o*	RhR14 *cv cB*	RhR15 *cv i*	RhR16 *ct d*	RhR17 *ct m*	RhR18 *ct o*	RhR19 *ct cB*
RRh1	[*disjoint disjoint*]																			
RRh2	[*meet disjoint*]																			
RRh3	[*overlap disjoint*]				3	3	3	3					2	2	2	2		2	2	2
RRh4	[*overlap meet*]			3	2	3	4	5					1	2	3	4		1	2	3
RRh5	[*overlap overlap*]			3	3		5	5							4	4				4
RRh6	[*overlap covers*]			3	4	5	4	5					1	2	3	4		1	2	3
RRh7	[*overlap contains*]			3	5	5	5	2					2	2	2	2		2	2	2
RRh8	[*covers contains*]																			
RRh9	[*contains contains*]																			
RRh10	[*equal contains*]																			
RRh11	[*coveredBy disjoint*]												2	2	2	2		2	2	2
RRh12	[*coveredBy meet*]			2	1		1	2				2		2	3	4		1	2	3
RRh13	[*coveredBy overlap*]			2	2		2	2				2	2		4	4				4
RRh14	[*coveredBy covers*]			2	3	4	3	2				2	3	4	2	4		1	2	3
RRh15	[*coveredBy contains*]			2	4	4	4	2				2	4	4	4	2		2	2	2
RRh16	[*inside disjoint*]																	3	3	3
RRh17	[*inside meet*]			2	1		1	2				2	1		1	2	3	2	3	4
RRh18	[*inside overlap*]			2	2		2	2				2	2		2	2	3	3		5
RRh19	[*inside covers*]			2	3	4	3	2				2	3	4	3	2	3	4	5	4
RRh20	[*inside contains*]			2	4	4	4	2				2	4	4	4	2	3	5	5	5
RRh21	[*inside equal*]			4	4	4	4	4				4	4	4	4	4	7	7	7	7
RRh22	[*inside coveredBy*]			4	3		3	4				4	3		3	4	7	6	3	6
RRh23	[*inside inide*]			4	4		4	4				4	4		4	4	7	7	3	7

Fig. 16. Crispness improvements (in absolute counts) for t_{RR_h} ; t_{R_hR} vs. t_{RR} ; t_{RR} (compositions without improvement left out; darker shading indicates stronger improvement).

Finding 8: From among the 266 compositions with crisper results, 27 (i.e., 10.2%) yield a *complete crispening*, that is a conversion from a universal composition to a unique composition. Complete crispenings occur only for compositions $\mathrm{RR}_h a ; \mathrm{R}_h \mathrm{R} b$ with $\pi(\mathrm{RR}_h a) = inside$ and $\pi(\mathrm{R}_h \mathrm{R} b) = contains$ (Figure 16). Resorting in these cases to the composition of their principal relations would incorrectly imply that these inferences are undetermined.

Finding 9: For all 266 compositions whose results are crisper, on average the crispness of each of these 266 compositions improves by 3.5 counts. Given that the highest possible improvement is seven (for a complete crispening), the average crispness improvement is 50%.

Finding 10: Compositions $\mathrm{RR}_h a\,;\mathrm{R}_h\mathrm{R}b$ are only subject to crispening if $\pi(\mathrm{RR}_h a) \in \{\mathit{overlap}, \mathit{coveredBy}, \mathit{inside}\}$ *and* $\pi(\mathrm{R}_h\mathrm{R}b) \in \{\mathit{overlap}, \mathit{covers}, \mathit{contains}\}$, yielding nine groups of compositions that feature crispenings (Figure 16). In these groups, the compositions with $\pi(\mathrm{RR}_h a) = \mathit{inside}$ and $\pi(\mathrm{R}_h\mathrm{R}b) = \mathit{contains}$ have the highest crispness improvements, both in absolute counts (319) as well as per composition (5.23, which corresponds to an average crispness improvement of 75%).

8 Conclusions

Most qualitative spatial reasoning has disregarded the inference constraints that cavities of geographic phenomena may impose, because their underlying models either explicitly exclude regions with holes from their domain or assume that the existence of a hole will have no impact on their topological inferences. To overcome these limitations, this paper studied systematically the topological relations of regions with a single hole, offering new insights for spatial reasoning over such regions:

While the 9-intersection captures eight topological relations between two regions, this number increases by 88% to 23 when one of the regions has a hole, yielding refinements of the eight region-region relations. Knowing the relation between a region and the generalized region implies a 63% chance (5 out of 8 relations) of uniquely identifying the complete relation between the two objects without any explicit reference to the relation with the hole.

The 23 relations' compositions over a common region with a hole show that these compositions form subsets—although not necessarily true subsets—of the results obtained from the compositions of regions without a hole. In 36% of the true subsets, the result is unique (i.e., a single relation). Approximately half of the compositions over a region with a hole yield fewer possible relations, with an 8% increase in the average crispness when compared to the results of compositions over a region without a hole. This decrease is due to a general trend of fewer results comprising five or more possibilities, in combination with an increase of the occurrence of results of fewer possibilities (four or less) and by a 10% increase of *complete crispness* (yielding a unique relation) among these improved results. This leads to an average crispness improvement of 50% for those results. These insights relate to people's reasoning performance, because relations that include regions with holes lead to a higher relative number of unique possible results.

These findings provide answers to the questions posed in the motivation: the more constrained composition inferences found for topological relations of a region with a hole are neither anomalies, nor do different inferences occur only in a single case. Since over 50% of the inferences with a hole are more refined than the corresponding inferences over regions without a hole, typically the reasoning over a region with a hole does differ from the well known topological inferences of regions without a hole,

Future work will pursue the derivation of complementary methods for similarity reasoning, such as the 23 relations' conceptual neighborhoods. Initial results indicate that this graph is an asymmetric extension of the graph for the eight region relations. We further intend to pursue the modeling of and inferences from binary topological relations between two regions, each with a hole. Finally, an interesting question for a larger theory of consistent qualitative reasoning across space and time is whether there are analog results to relations over regions with holes in the temporal domain, namely for intervals with gaps.

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