Representing Curves

Foley & Van Dam, Chapter 11

Representing Curves

- Motivations
- Techniques for Object Representation
- Curves Representation
- Free Form Representation
- Approximation and Interpolation
- Parametric Polynomials
- Parametric and Geometric Continuity
- Polynomial Splines
  - Hermite Interpolation

3D Objects Representation

- Solid Modeling attempts to develop methods and algorithms to model and represent real objects by computers

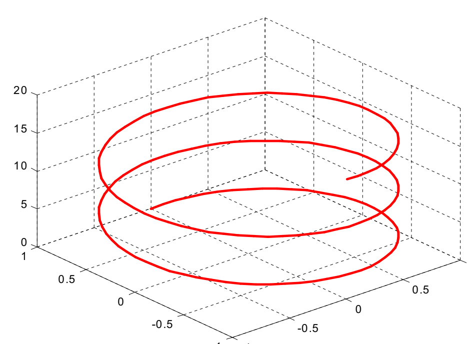
Objects Representation

- **Three types of objects in 3D:**
  - 1D curves
  - 2D surfaces
  - 3D objects
- **We need to represent objects when:**
  - Modeling of existing objects (3D scan)
    - modeling is not precise
  - Modeling a new object "from scratch" (CAD)
    - modeling is precise
    - interactive sculpting capabilities

General Techniques

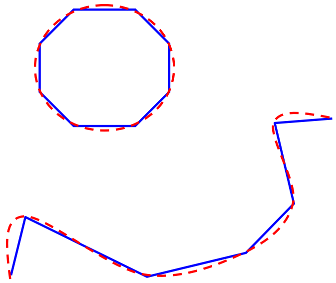
- **Primitive Based:**
  A composition of "simple" components
  - Not precise
  - Efficient and simple
- **Free Form:**
  Global representation, curved manifolds
  - Precise
  - Complicated
- **Statistical:**
  Modeling of objects generated by statistical phenomena, such as fog, trees, rocks

Curves Representation



Primitive Based Representation

- **Line segments:** A curve is approximated by a collection of connected line segments



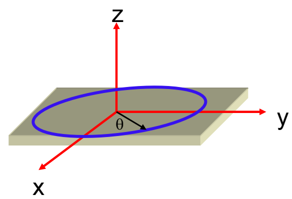
Free Form Representations

- **Explicit form:** $z = f(x, y)$
  - $f(x,y)$ must be a function
  - Not a rotation invariant representation
  - Difficult to represent vertical tangents
- **Implicit form:** $f(x, y, z) = 0$
  - Difficult to connect two curves in a smooth manner
  - Not efficient for drawing
  - Useful for testing object inside/outside
- **Parametric:** $x(t), y(t), z(t)$
  - A mapping from $[0,1] \rightarrow R^3$
  - Very common in modeling

Free Form Representations

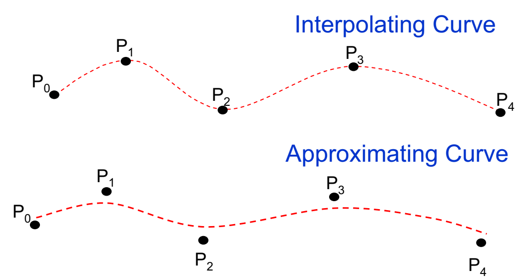
Example: A Circle of radius R

- Implicit:
  $x^2 + y^2 + z^2 - R^2 = 0 \;\;\&\;\; z = 0$
- Parametric:
  $x(\theta) = R\cos(\theta)$
  $y(\theta) = R\sin(\theta)$
  $z(\theta) = 0$



Approximated vs. Interpolated Curves

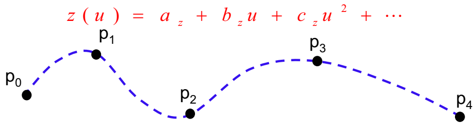
- Given a set of *control points* P_i known to be on the curve, find a parametric curve that interpolates/approximates the points



Parametric Polynomials

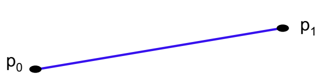
- For interpolating n points we need a polynomial of degree n-1

$$x(u) = a_x + b_x u + c_x u^2 + \cdots$$
$$y(u) = a_y + b_y u + c_y u^2 + \cdots$$
$$z(u) = a_z + b_z u + c_z u^2 + \cdots$$



- Example: Linear polynomial. For interpolating 2 points we need a polynomial of degree 1

$$x(u) = a_x + b_x u$$
$$y(u) = a_y + b_y u$$
$$z(u) = a_z + b_z u$$



Example: Linear Polynomial

- The geometrical constraints for x(u) are:

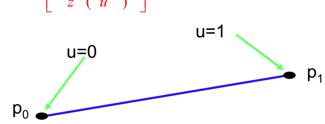
$$x(0) = a_x = P_0^x \quad ; \quad x(1) = a_x + b_x = P_1^x$$

- Solving the coefficients for x(u) we get:

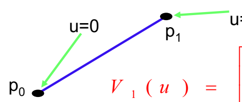
$$a_x = P_0^x \quad ; \quad b_x = P_1^x - P_0^x$$
$$\Rightarrow \quad x(u) = P_0^x + (P_1^x - P_0^x)\, u$$

- Solving for [x(u) y(u) z(u)] we get:

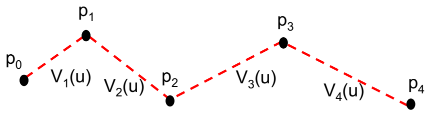
$$V(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} = P_0 + (P_1 - P_0)\, u$$



Example: Linear Polynomial



$$V_1(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} = P_0 + (P_1 - P_0)\ u$$

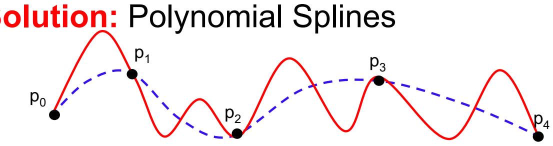


$$V(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = \sum_i \hat{V}_i(u)$$

$$\text{where} \qquad \hat{V}_i(u) = \begin{cases} V_i(u) & \text{if} \quad u \in [u_i, u_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

Parametric Polynomials

- Polynomial interpolation has several disadvantages:
  - Polynomial coefficients are geometrically meaningless
  - Polynomials of high degree introduce unwanted wiggles
  - Polynomials of low degree give little flexibility
- **Solution:** Polynomial Splines



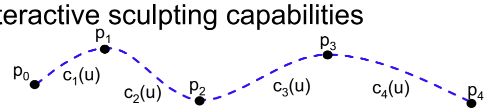
Polynomial Splines

- Piecewise, low degree, polynomial curves, with continuous joints

$$C(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = \sum_i \hat{C}_i(u)$$

$$\text{where} \qquad \hat{C}_i(u) = \begin{cases} C_i(u) & \text{if} \quad u \in [u_i, u_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

- Advantages:
  - Rich representation
  - Geometrically meaning coefficients
  - Local effects
  - Interactive sculpting capabilities

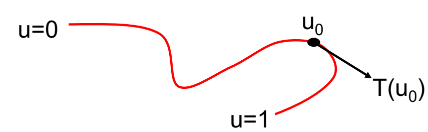


Tangent Vector

- Let $V(u)=[x(u), y(u), z(u)]$, $u \to [0,1]$ be a continuous univariate parametric curve in R^3
- The tangent vector at u_0, $T(u_0)$, is:

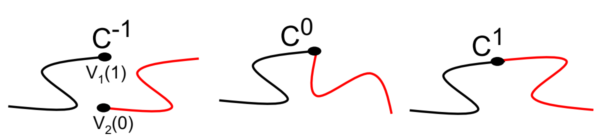
$$\vec{T}(u_0) = V'(u_0) = \left.\frac{dV(u)}{du}\right|_{u=u_0} = \left[\frac{dx}{du}\ \frac{dy}{du}\ \frac{dz}{du}\right]_{u=u_0}$$

- V(u) may be thought of as the trajectory of a point in time
- In this case, $T(u_0)$ is the instantaneous velocity vector at time u_0



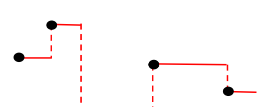
Parametric Continuity

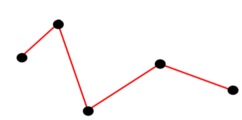
- Let $V_1(u)$ and $V_2(u)$, $u \to [0,1]$, be two parametric curves
- Level of parametric continuity of the curves at the joint between $V_1(1)$ and $V_2(0)$:
  - C^{-1}: The joint is discontinuous, $V1(1) \neq V2(0)$
  - C^0: Positional continuos, $V_1(1)=V_2(0)$
  - C^1: Tangent continuos, C^0 & $V'_1(1)=V'_2(0)$
  - C^k, k>0: Continuous up to the k-th derivative, $V_1^{(j)}(1)=V_2^{(j)}(0), \quad 0 \leq j \leq k$



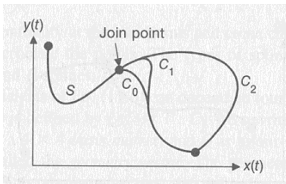
Geometric Continuity

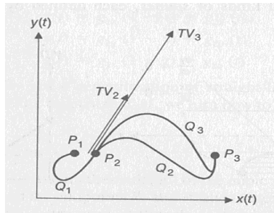
- In computer aided geometry design, we also consider the notion of *geometric continuity*:
  - G^{-1}, G^0: Same as C^{-1} and C^0
  - G^1: Same tangent direction: $V'_1(1)=\alpha V'_2(0)$
  - G^k: All derivatives up to the k-th order are proportional
- Given a set of points {pi}:
  - A piecewise constant interpolant is C^{-1}
  - A piecewise linear interpolant is C^0





Parametric and Geometric Continuity

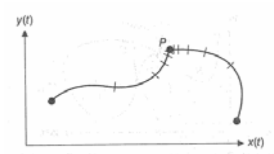




- S-C_0 is C^0
- S-C_1 is C^1
- S-C_2 is C^2

- Q_1-Q_2 both C^1 and G^1
- Q_1-Q_3 is G^1 but not C^1

- In general, C^i implies G^i (not vice versa)
- Exception when the tangents are zero



Parametric Cubic Curves

- Cubic polynomials defining a curve in R^3 have the form:

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$
$$y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$$
$$z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$$

Where u is in [0,1]. Defining:

$$U^T(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

The curve can be rewritten as:

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = V^T(u) = U^T(u)\, Q$$

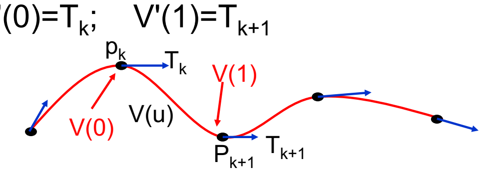
Parametric Cubic Curves

- The coefficients Q are unknown and should be determined
- For this purpose we have to supply 4 geometrical constraints
- Different types of constraints define different types of Splines

Hermite Curves

- Assume we have n control points $\{p_k\}$ with their tangents $\{T_k\}$
- W.L.O.G. V(u) represents a parametric cubic function for the section between p_k and p_{k+1}
- For V(u) we have the following geometric constraints:

$$V(0)=p_k; \quad V(1)=p_{k+1}$$
$$V'(0)=T_k; \quad V'(1)=T_{k+1}$$



Hermite Curves

Since

$$V^T(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} Q$$

we have that

$$(V')^T(u) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} Q$$

We can write the constraints in a matrix form:

$$G = MQ \quad \Rightarrow \quad \underbrace{\begin{bmatrix} p_k \\ p_{k+1} \\ T_k \\ T_{k+1} \end{bmatrix}}_{G} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}}_{M} Q$$

And thus $V^T(u) = U^T(u)Q = U^T(u)M^{-1}G$

Where

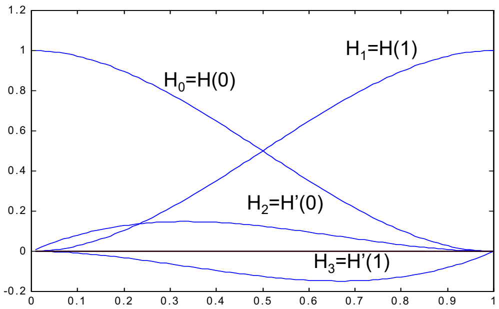
$$M^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hermite Curves

$$V(u) = \underbrace{[u^3 \quad u^2 \quad u \quad 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\text{Blending functions}} \underbrace{\begin{bmatrix} p_k \\ p_{k+1} \\ T_k \\ T_{k+1} \end{bmatrix}}_{\text{Geometry matrix}}$$

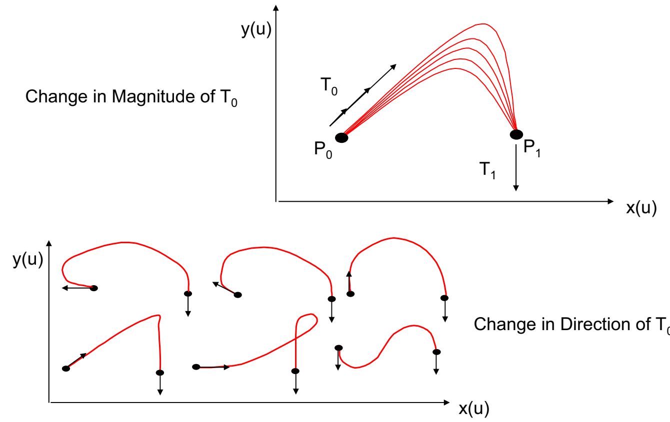
$$V(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_k \\ p_{k+1} \\ T_k \\ T_{k+1} \end{bmatrix} = \begin{bmatrix} H_0(u) \\ H_1(u) \\ H_2(u) \\ H_3(u) \end{bmatrix}^T \begin{bmatrix} p_k \\ p_{k+1} \\ T_k \\ T_{k+1} \end{bmatrix}$$

Hermite Curves



Hermite Blending Functions

Hermite Curves



Hermite Curves

Properties:

- The Hermite curve is composed of a linear combinations of tangents and locations (for each u)
- Alternatively, the curve is a linear combination of Hermite basis functions (the matrix M)
- It can be used to create geometrically intuitive curves
- The piecewise interpolation scheme is C^1 continuous
- The blending functions have local support; changing a control point or a tangent vector, changes its local neighborhood while leaving the rest unchanged

Hermite Curves

- **Main Drawback:**

Requires the specification of the tangents
This information is not always available