Ling130: Semantics

1/22/10 Lecture with Jessica Moszkowicz

- **Goal for today**
  - Feel comfortable manipulating logical symbols so that, when we add complexity to the logical systems with quantifiers, etc., you'll still have these basic natural deduction rules and strategies in your arsenal

- **Propositional Logic**
  - What is a proposition? – *Anything that can be true or false.*
    - Ex. "Elana is a cute, chubby baby."
    - This whole thing is one proposition. Clearly, that hides a lot of the internal structure of the sentence that we will want to get at.
      - That's where Predicate Logic (First Order Logic) comes in, but, for now, we're content to hide all of the internal stuff and just call this entire sentence P.
  - What are logical connectives? – *They hold* ***atomic sentences*** *like P and Q together to make* ***complex sentences****.*
    - We basically know what each connective "means", but that doesn't matter at all for natural deduction.
      - Natural deduction is just matching symbols and pushing symbols around! It doesn't matter in the slightest what the propositions or connectives stand for.
      - Still, it's helpful to motivate the rules of ND by thinking about the intended meanings of the connectives.
    - Our connectives:
      - Conditional: $\rightarrow$
      - Negation: $\neg$
      - And: $\wedge$
      - Or: $\vee$
  - What is the main connective? (a.k.a. What kind of sentence is it?) – *The connective that holds the whole sentence together.*
    - $P\rightarrow Q$ – conditional
    - $P\wedge Q$ – conjunction
    - $P\vee Q$ – disjunction
    - $\neg P$ – negation
    - $(P\wedge Q)\rightarrow\neg(R\rightarrow(S\vee T))$ – conditional

- Why is the main connective important?
  - For each connective (basically), there are two rules of inference: ***Elimination and Introduction***
  - Elimination rules USE sentences of a particular kind to break them apart
  - Introduction rules CREATE sentences of a particular kind either to prove the goal of the proof or to help move the proof along

## ❖ Natural Deduction for Propositional Logic

- *Conditional (→) rules*

| P | Q | P→Q |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

  - What does the truth table tell us? – *Conditionals are only false when the antecedent is true, but the consequent is false.*
    - Keep that in mind as we use the rules, but remember that this is just motivation.
  - *Conditional Elimination (modus ponens)*
    - $\frac{\phi \to \psi, \phi}{\psi}$ or P→Q, P ⊢ Q
    - →E requires a conditional and its antecedent
    - Basic Proofs: Line Number, Formula, Reason
      - ♦ 1 P→Q Basic Assumption

        2 P Basic Assumption ⊢ Q

        3 Q 1,2 →E
  - *Conditional Introduction(conditional proof)*
    - →I says: If you assume the antecedent is true and are able to prove the consequent, then you can infer the whole conditional and discharge the assumption since it doesn't matter if the antecedent is true or false. (remember the truth table)
    - Example: P→Q, Q→R ⊢ P→R

| | | | |
|---|---|---|---|
| ♦ 1 | P→Q | Basic Assumption | |
| 2 | Q→R | Basic Assumption | ⊢ P→R |
| 3 | P | Assume (for →I) | ⊢ R |
| 4 | Q | 1,2 →E | |
| 5 | R | 2,4 →E | |
| 6 | P→R | 3-5 →I | |

- The Deal with Assumptions
    - You can always assume whatever you want in your proof, but you have to <u>discharge</u> it before the proof ends. Otherwise, your proof might depend on something that isn't true.
    - Only assume something if you know how to discharge it!
- First bit of strategy:
    - $\rightarrow$I is a way to prove conditionals. If your conclusion is a conditional, a good way to get started is to assume the antecedent and then try to prove the consequent.

- Conditional Examples
    - $P\rightarrow(P\rightarrow Q), P \vdash Q$
    - $P\rightarrow Q \vdash (Q\rightarrow R)\rightarrow(P\rightarrow R)$

➢ *Negation ($\neg$) Rules, part 1*

- *Double Negation*
    - $\neg\neg P \vdash P$ and $P \vdash \neg\neg P$
    - This is a silly rule! It's ok with me if you ignore it completely.
- *Modus Tollens*
    - $P\rightarrow Q, \neg Q \vdash \neg P$
    - This rule is like a fancy version of $\rightarrow$E.
- Example: $\neg P\rightarrow\neg Q, Q \vdash P$
    - | | | | |
      |---|---|---|---|
      | 1 | $\neg P\rightarrow\neg Q$ | Basic Assumption | |
      | 2 | $Q$ | Basic Assumption | $\vdash P$ |
      | 3 | $\neg\neg Q$ | 2 DN | |
      | 4 | $\neg\neg P$ | 1,3 MT | |
      | 5 | $P$ | 4 DN | |
    - I don't like this proof because it doesn't seem to follow a good strategy. We'll see in a bit that we could easily have proved P with a proof by negation and then we wouldn't have to do any of this crazy backwards MT stuff. Just my opinion!

➢ *Conjunction ($\wedge$) Rules*

- *$\wedge$ Introduction*
    - $P, Q \vdash P\wedge Q$
    - You can always take two sentences from your proof and $\wedge$ them together.
    - Use $\wedge$I to prove a conjunction either as your conclusion or on your way to your conclusion.
- *$\wedge$ Elimination*

- $P \wedge Q \vdash P$ and $P \wedge Q \vdash Q$
- You can always break apart a conjunction to get at its parts

- Examples
  - $(P \wedge Q) \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$
    - Assume P, assume Q, …
  - $P \vdash Q \rightarrow (P \wedge Q)$
  - $(P \rightarrow Q) \wedge (P \rightarrow R) \vdash P \rightarrow (Q \wedge R)$

- *Disjunction ($\vee$) Rules*
  - *$\vee$ Introduction*
    - $P \vdash P \vee Q$
    - Use $\vee$I to prove a disjunction
    - It doesn't matter what 'Q' is; it can be totally new to the proof.
  - *$\vee$ Elimination (proof by cases)*
    - $\vee$E is a way to use a disjunction
    - It involves two mini-proofs:
      - $P \vee Q \vdash R$
      - Assume P, prove R.
      - Assume Q, prove R.
      - Explanation:
        - You know from $P \vee Q$ that either P is true or Q is true (or they're both true).
        - So, if you can prove R in both cases, then you know R must be true.
    - Example: $P \rightarrow R$, $Q \rightarrow R$, $P \vee Q \vdash R$
      - 

| | | |
|---|---|---|
| 1 | $P \rightarrow R$ | Basic Assumption |
| 2 | $Q \rightarrow R$ | Basic Assumption |
| 3 | $P \vee Q$ | Basic Assumption $\vdash R$ |
| 4 | P | Assume (for $\vee$E) $\vdash R$ |
| 5 | R | 1,4 $\rightarrow$E |
| 6 | Q | Assume (for $\vee$E) $\vdash R$ |
| 7 | R | 2,6 $\rightarrow$E |
| 8 | R | 3, 4-5, 6-7 $\vee$E |

  - Disjunction Example
    - $P \rightarrow Q$, $R \rightarrow S \vdash (P \vee R) \rightarrow (Q \vee S)$

- *Negation ($\neg$) Rules, Part 2*
  - *RAA (Reductio ad Absurdum) ($\neg$I) (proof by negation)*
    - Let's say you assume P and from that assumption you are able to prove a contradiction ($Q \wedge \neg Q$). That means that your assumption can't be true, so you've proven $\neg P$.
    - Example: $P \rightarrow Q$, $P \rightarrow \neg Q \vdash \neg P$
      - 

| | | |
|---|---|---|
| 1 | $P \rightarrow Q$ | Basic Assumption |
| 2 | $P \rightarrow \neg Q$ | Basic Assumption $\vdash \neg P$ |
| 3 | $P$ | Assume (for RAA) $\vdash$ # |
| 4 | $Q$ | 1,3 $\rightarrow$E |
| 5 | $\neg Q$ | 2,4 $\rightarrow$E |
| 6 | $Q \wedge \neg Q$ | 4,5 $\wedge$I |
| 7 | $\neg P$ | 3-6 RAA |

    - RAA can be use to prove a negation, but it can also be used as a general strategy for proving anything.
    - Still, it's not always the best strategy, so I like to consider it a rule of last resort unless you're actually trying to prove a negation.

- **Proof Strategy**
  - Step 1: Check that there's no obvious way to do the proof.
  - Step 2: Look at the basic assumptions for a disjunction. If there is one, immediately start using $\vee$E.
  - Step 3: Identify what kind of sentence the conclusion is and then do the following:
    - Conditional – use $\rightarrow$I (assume the antecedent and prove the consequent)
    - Conjunction – use $\wedge$I (Prove each conjunct)
    - Disjunction – use $\vee$I (Prove either disjunct)
    - Negation – use RAA (Assume the negate and prove a contradiction)
    - Atomic – check again that there's no quick solution. If there really isn't, then use RAA (Assume the negation of the conclusion and prove a contradiction)
  - Step 4: If all else fails, try RAA by negating the desired conclusion and then proving a contradiction.