A General Coarse-Graining Framework for Studying Simultaneous Inter-Population Constraints Induced by Evolutionary Operations

Keki Burjorjee & Jordan B. Pollack

DEMO Lab
Computer Science Department
Brandeis University

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Assume there is some population in some evolutionary system (GA, GP, ES, etc.)

- Call it $p$

Suppose that either selection or variation is applied to $p$

- Call the resulting population $q$

What are the constraints on the composition of $q$?
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- Can $q$ be any population?
- (Clearly) No. Its composition will be constrained by
  - The composition of $p$
  - Which evolutionary operation $\mathcal{W}$ is

**Terminology**

- $p$: pre-operative population of $\mathcal{W}$
- $q$: post-operative population of $\mathcal{W}$

An inter-population constraint is a constraint that an evolutionary operation induces between any pre and post operative populations.
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Introduction

Elements of the General Coarsegraining Framework
Inter-Pop. Constraints Induced by Selection And Variation
Reduction to a Framework for Studying Population Marginals

Inter-Population Constraints

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The Schema Theoretic approach is to:

1. Define a grammar for specifying schemata, then
2. Derive a schema theorem that describes how evolutionary operations alter the frequencies of schemata

The conditions for applicability of each schema theorem are very strict.

Each theory applicable only when

- Genotypes are of a particular datastructure
- Particular variation operators are used
- Schemata are defined in a particular way

The results of each schema theorem do not carry over when any of these change.
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A Different Technique for Deriving Inter-Population Constraints

- Form Invariant Commutation: a different technique for deriving inter-population constraints
- Uses coarse-grainings
- A coarse-graining is just a function from the genotype set to some set
  - e.g. if $G$ is a genotype set and $K$ is some set then $\beta : G \rightarrow K$ is a coarse-graining
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An Abstract Coarse-Graining Framework

- Present a framework that uses form invariant commutation to obtain theorems about the inter-population constraints induced by evolutionary operations.
- Framework is *abstract* in that unlike various schema theories, applicability of theorems in this framework is *not* limited by:
  - a specific genotypic datastructure, or
  - specific variation operators, or
  - a specific grammar for defining schemata.
- Rather, applicability of results is limited by whether an *abstract relationship* exists between the variation operators and coarse-grainings used.
  - Relationship is called *Ambivalence*. 
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Benefits of this Framework

- Given some
  - fitness proportional evolutionary system, and some
  - coarse-graining

  Easy to determine whether this framework is useful for deriving inter-population constraints
  - Just check if the ambivalence relationship holds

- Determination of whether schema analysis is useful typically proceeds by “trying to do the schema analysis”

- Once it is determined that this framework is useful, burden of analysis is greatly reduced
  - The “grunt work” is done by the “machinery” of the framework

- Framework accommodates multi-parent variation operators in a natural way

- For the set of fixed length bitstring GAs with ‘common’ variation operations
  - General framework reduces to a specific framework for analyzing the effect of one evolutionary step on the marginal distributions of populations
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Use a transmission functions (Altenberg 1994) to model any stochastic variation function

- that takes $n$ genotypes as parents, and
- produces 1 genotype as a child

Example: $T(g|g_1, \ldots, g_n)$ is the probability that some variation operation produces $g$ as a child given parents $g_1, \ldots, g_n$. 
Use a transmission functions (Altenberg 1994) to model any stochastic variation function that takes $n$ genotypes as parents, and produces 1 genotype as a child.

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Modeling Populations and Operations on Populations

- Populations modeled as real valued distributions over the genotype set.
  - Distribution values sum to 1
- Evolutionary operations are modeled as \textit{parameterized} mathematical operators
  - Take genotypic distributions as input and produce genotypic distributions as output
  - Parameter objects used by the operator in the calculation of output
Modeling the Effect of Variation on Populations

- Effect of any variation function modeled by $T$ on some population $p$ is given by the variation operator $\mathcal{V}_T$

If $p' = \mathcal{V}_T(p)$, then $p'$ is as follows: For any genotype $g$,

$$p'(g) = \sum_{(g_1, \ldots, g_m) \in \prod_1^m G} T(g|g_1, \ldots, g_m) \prod_{i=1}^m p(g_i)$$
Effect of fitness proportional selection on some population $p$ using any fitness function $f$ given by the selection operator $S_f$

If $p' = S_f(p)$, then $p'$ is as follows: For any genotype $g$,

$$p'(g) = \frac{f(g)p(g)}{\mathcal{E}_f(p)}$$

where $\mathcal{E}_f$ is the weighted average fitness of $p$
Coarse-graining Terminology and Notation

For any coarsegraining \( \beta : G \to K \):
- Call co-domain \( K \) the \( \beta \)-theme set
- Call the elements of \( K \) \( \beta \)-themes

For any \( g \in G, k \in K \) such that \( \beta(g) = k \), say that \( g \) \( \beta \)-instantiates \( k \):
- \( \langle k \rangle_\beta \) denotes the set of all \( g \in G \) that \( \beta \)-instantiate \( k \)
- Call \( \langle k \rangle_\beta \) the \( \beta \)-theme class of \( k \)
Projection Operator

- Let $\beta : G \to K$ be a coarsegraining.
- A projection operator $\Xi_{\beta}$ ‘projects’ a distribution $p_G$ over $G$ ‘through’ $\beta$ to create a distribution $p_K = \Xi_{\beta}(p_G)$ over the theme set.

For any $k \in K$,

$$p_K(k) = \sum_{g \in (k)_\beta} p(g)$$
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A Technique for Obtaining Inter-Population Constraints

Form Invariant Commutation

- $\beta : G \rightarrow K$ some coarsegraining
- $\mathcal{W}$ a parameterizable operator parameterized by some object $x$ such that
- for any population $p_G$ such that

Then we've obtained a single inter population constraint for the operator $\mathcal{W}_x$

Call $y$ the quotient parameter

Okay for the $y$ quotient parameter to depend on $p_G$

- As long as the nature of this dependence is well understood

A single inter-population constraint is good . . .
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\begin{align*}
  p_G & \xrightarrow{\mathcal{W}_x} p'_G \\
  p_K & \xrightarrow{\mathcal{W}_y} p'_K \\
\end{align*}
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- Then we've obtained a single inter population constraint for the operator $\mathcal{W}_x$
- Call $y$ the *quotient parameter*
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\beta & : G \to K \\
\mathcal{W} & : \text{a parameterizable operator parameterized by } x \\
\Xi & : \text{some coarsegraining}
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More inter-population constraints give more information.

- if $\beta_1, \ldots, \beta_n$ induce different partitions over the genotype set
- Call these *simultaneous* constraints
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Selectional Constraints Theorem

For any coarse-graining $\beta : G \rightarrow K$ and any population $p_G$,

$$p_G \xrightarrow{S_f} p_G'$$

$\Xi_\beta \xrightarrow{p_K} \leftarrow p_K' \xleftarrow{\Xi_\beta}$

$F(\beta, p_G) : K \rightarrow \mathbb{R}^+$ is called the $\beta$-theme fitness function of $p_G$

- It assigns to each theme $k \in K$ the weighted average fitness of all genotypes that $\beta$-instantiate $k$
- $F(\beta, p_G)$ depends on $p_G$, but that’s okay since we understand the (simple) nature of this dependence
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\[
\Xi_\beta \xrightarrow{\cong} \xi_\beta
\]

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Simultaneous Selectional Constraints

▶ Result of the selection constraints theorem is valid for any coarse-graining
▶ So its easy to obtain simultaneous selectional constraints
▶ For any coarsegrainings $\beta_1, \ldots, \beta_n$, and any population $p_G$, 

\[
p_G \xrightarrow{\Xi_{\beta_1}} p_{\beta_1} K_1 \xrightarrow{S_{F(\beta_1, p_G)}} p'_{K_1} \\
\vdots \\
p_G \xrightarrow{\Xi_{\beta_n}} p_{\beta_n} K_n \xrightarrow{S_{F(\beta_n, p_G)}} p'_{K_n}
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What about Variation?

- Inter-population constraints easy to derive for the selection operator
  - *Any* coarse-graining can be used
- Matters not so simple in the case of variation
- Introduce a relationship between a transmission function and a coarse-graining called *ambivalence*
- When ambivalence is satisfied derivation of an inter-population variational constraint using the coarse-graining is possible
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Ambivalence by Example

An Ambivalent 2-parent transmission function $T$

Say that $T$ is ambivalent under $\beta$
Ambivalence by Example

An Ambivalent 2-parent transmission function $T$

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Say that $T$ is *ambivalent* under $\beta$. 

An Ambivalent 2-parent transmission function $T$
Ambivalence by Example

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Projection of an Ambivalent Transmission function

- $T$ an ambivalent under some coarse-graining $\beta : G \rightarrow K$,
- $\beta$-projection of $T$, denoted $T^\beta$, is a transmission function over $K$.
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An Example of Ambivalence
Variational Constraints Theorem

- Given
  - Coarse-graining $\beta : G \rightarrow K$
  - And transmission function $T$
- Such that $T$ is ambivalent under $\beta$
- For any population $p_G$

\[\begin{align*}
p_G \xrightarrow{\nu_T} p'_G \\
\Xi_\beta \xrightarrow{T_\beta} p_K \xrightarrow{\nu_T} p'_K \\
\Xi_\beta
\end{align*}\]
Let \( G \) be a set of fixed length bitstrings

- Schema partitioning: A function that maps any genotype to its values at some fixed set of locii.
  - Example: \( \beta_{1,3} \) maps any genotype to its 1\(^{st}\) and 3\(^{rd}\) bits
- Schema partitionings induce schema partitions on the genotype set
- Example continuation: let \( G \) be the set of bitstrings of length 8
  - Then \( \beta_{1,3} \) induces the schema partition #**#***** over \( G \)
Schema Partitionings

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Schema partitionings induce schema partitions on the genotype set.

Example continuation: let $G$ be the set of bitstrings of length 8.

- Then $\beta_{1,3}$ induces the schema partition #**####** over $G$.
If genotypes are bitstrings of fixed length
If the variation operation consists of some combination of
  ▶ n-point crossover
  ▶ Uniform crossover
  ▶ Canonical mutation
    ▶ Probability of mutation is constant for each bit
Variation is ambivalent under any schema partitioning
Any schema partitioning can be used to derive a variational constraint
Simultaneous Constraints for GA Variation Operations Using Schema Partitionings

For any schema partitionings $\beta_1, \ldots, \beta_n$, and any common GA variation operators represented by $T$
In a common GA, any schema partitioning \( \beta \) gives us both a selectional constraint and a variational constraint. ‘Concatenating’ these constraints we obtain:

\[
p_G \xrightarrow{S_f} p'_G \xrightarrow{\nu_T} p''_G
\]

Projection of a population through a schema partitioning is essentially a marginalization operation. Thus for a GA with common variation operations the general framework reduces to a specific framework for studying the effect of one evolutionary step on population marginals.
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\[
p_G \xrightarrow{S_f} p'_G \xrightarrow{\nu_T} p''_G
\]

\[
p_K \xrightarrow{S_{F(\beta, p_G)}} p'_K \xrightarrow{\nu_{\beta \rightarrow T}} p''_K
\]

Projection of a population through a schema partitioning is essentially a marginalization operation. Thus for a GA with common variation operations the general framework reduces to a specific framework for studying the effect of one evolutionary step on population marginals.