

# Ling 130 Notes: A guide to Natural Deduction proofs

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## 1 Strategy summary

Given a set of premises,  $\Delta$ , and the goal you are trying to prove,  $\Gamma$ , there are some simple rules of thumb that will be helpful in finding ND proofs for problems. These are not complete, but will get you through our problem set. Here goes:

Given  $\Delta$ , prove  $\Gamma$ :

1. Apply the premises,  $p_i$  in  $\Delta$ , to prove  $\Gamma$ .
2. If you need to use a premise which is a disjunction ( $\vee$ -premise), then apply Disjunction Elimination ( $\vee E$ ) rule and prove  $\Gamma$  for each disjunct.
3. Otherwise, work backwards from the type of goal you are proving:
  - (a) If the goal  $\Gamma$  is a conditional ( $A \rightarrow B$ ), then assume  $A$  and prove  $B$ ; use arrow-introduction (Conditional Introduction,  $\rightarrow I$ ) rule.
  - (b) If the goal  $\Gamma$  is a negative ( $\neg A$ ), then assume  $\neg\neg A$  (or just assume  $A$ ) and prove contradiction; use Negation Introduction ( $\neg I$ ) rule.
  - (c) If the goal  $\Gamma$  is a conjunction ( $A \wedge B$ ), then prove  $A$  and prove  $B$ ; use Conjunction Introduction ( $\wedge I$ ) rule.
  - (d) If the goal  $\Gamma$  is a disjunction ( $A \vee B$ ), then prove one of  $A$  or  $B$ ; use Disjunction Introduction ( $\vee I$ ) rule.

## 2 Using rules for conditionals

- **Conditional Elimination** (modus ponens) ( $\rightarrow E$ )

$\phi \rightarrow \psi, \phi \vdash \psi$

This rule requires a conditional and its antecedent. All basic proofs require, on each line, line number, the formula that you deduce on that line, and the reason why you can deduce that - namely, the rule and the premises that prove the formula.

- (1) Example:  $p \rightarrow q, p \vdash q$
- |   |                   |                           |
|---|-------------------|---------------------------|
| 1 | $p \rightarrow q$ | basic assumption (given)  |
| 2 | $p$               | basic assumption          |
| 3 | $q$               | $\rightarrow E$ lines 1,2 |

- **Conditional Introduction** (conditional proof) ( $\rightarrow I$ )

This rule says: If you assume the antecedent is true and are able to prove the consequent, then you can infer the whole conditional and discharge the assumption since it doesn't matter if the antecedent is true or false. (remember the truth table!)

(2) Example:  $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$

1	$p \rightarrow q$	basic assumption
2	$q \rightarrow r$	basic assumption (want to prove $p \rightarrow r$ )
3	$p$	additional assumption (for $\rightarrow I$ (discharge after $\vdash r$ ))
4	$q$	$\rightarrow E$ , lines 1,3
5	$r$	$\rightarrow E$ , lines 2,4
6	$p \rightarrow r$	$\rightarrow I$ , lines 3-5

See above for the primary use of this rule - in proving conditionals!

### 3 The Deal with Assumptions

- You can always assume whatever you want in your proof, but you have to discharge it before the proof ends.  
Otherwise, your proof might depend on something that isn't true.
- Only assume something if you know how to discharge it!

### 4 More negation rules!

- In the slides, we only mentioned the Double Negation Elimination rule. As with other rules, there is also an Introduction rule, given in the book.

With both the  $\neg\neg I$  and  $\neg\neg E$  rules, you can now add or remove double negation in front of any proposition.

Last class, we used the ND rules from the slides to construct a proof of  $p \rightarrow q, \neg q \vdash \neg p$

- There is, in fact, a rule that proves that in one step. It's like a fancy version of  $\rightarrow E$
- It's called *Modus Tollens*  
Often, this rule seems a bit crazy and backwards, and as you noticed from our proving it, it's not exactly necessary. But it can be useful sometimes, and it has a fancy Latin name, so you should know it.

(3) Example:  $\neg p \rightarrow \neg q, q \vdash p$

1	$\neg p \rightarrow \neg q$	basic assumption
2	$q$	basic assumption (goal: $\vdash p$ )
3	$\neg\neg q$	Double Negation Introduction, line 2
4	$\neg\neg p$	Modus Tollens, lines 1, 3
5	$p$	Double Negation Elimination, line 4

Proof by negation would have been more intuitive here, actually.

- Remember *Reduction ad Absurdum* ( $\neg I$ )?

Let's say you assume  $p$  and from that assumption you're able to prove a contradiction. That means that your assumption can't be true, so you've proved  $\neg p$ .

(4) Example  $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$

1	$p \rightarrow q$	basic assumption
2	$p \rightarrow \neg q$	basic assumption (goal: $\vdash \neg p$ )
3	$p$	Assume (for RAA to prove $\perp$ )
4	$q$	$\rightarrow E$ lines 1,3
5	$\neg q$	$\rightarrow E$ lines 1,4
6	$q \wedge \neg q$	$\wedge I$ lines 4,5
7	$\perp$	$\neg E$ line 6
7	$\neg p$	RAA, 3-6

- RAA can be used to prove a negation, but also as a general strategy for proving anything.
- Still, it's not always the best strategy, so please use it as a last resort unless you're actually trying to prove a negation.

## 5 Disjunction and proof by cases

- Remember, that  $\vee E$  rule involves two mini-proofs:

- You're trying to prove  $r$  from  $p \vee q$
- Assume  $p$ , prove  $r$
- Assume  $q$ , prove  $r$
- Discharge both  $p$  and  $q$  and conclude  $r$

The two mini-proofs are needed because you know from  $p \vee q$  that either  $p$  is true or  $q$  is true (or they're both true) - remember the truth table!

So, if you can prove  $r$  in both cases, then you know  $r$  must be true.

(5) Example:  $p \rightarrow r, q \rightarrow r, p \vee q \vdash r$

1	$p \rightarrow r$	basic assumption
2	$q \rightarrow r$	basic assumption
3	$p \vee q$	basic assumption (goal: $\vdash r$ )
4	$p$	Assume (for $\vee E$ to prove $r$ )
5	$r$	$\rightarrow E$ lines 1,4
6	$q$	Assume (for $\vee E$ to prove $r$ )
7	$r$	$\rightarrow E$ lines 2,6
8	$r$	$\vee E$ lines 3,4-5,6-7

Note that when we give the lines, there could be many more than two involved!

(5) Example:  $p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$

## 6 Proof Strategy: recap

- Step 1: Check that there's no obvious way to do the proof.
- Step 2: Look at the basic assumptions to see if there's a disjunction. If there is one, immediately start using  $\vee E$ .
- Step 3: Identify what kind of sentence the conclusion is and then do the following:
  - Conditional: use  $\rightarrow I$  (assume antecedent and prove the consequent)
  - Conjunction: use  $\wedge I$  (prove each conjunct)
  - Disjunction: use  $\vee I$  (prove either disjunct)
  - Negation: use RAA (assume the negation and prove a contradiction)
  - Atomic: check again that there's no quick solution. If there really isn't, then use RAA (assume the negation of the conclusion and prove a contradiction)
- Step 4: If all else fails, try RAA by negating the desired conclusion and then proving a contradiction.

## References

- Jessica Moszkowicz's lecture notes in *Ling 130: Semantics*, 1/22/2010, Brandeis University
- James Pustejovsky's ND tips in *Ling 130: Semantics*, Spring 2010, Brandeis University