1 The Type of a Quantified Expression

Before Generalized Quantifiers were introduced, we only considered NPs that were proper names, hence interpreted as constants in our model. But we need to be able to refer to individuals by description, and not by name. For example, we need some way to talk about the following NPs: a man, every course, some professor. We sort of know what we want the formula to look like when they’re combined with predicates:

(1) a. a man walked. $\exists x[\text{man}(x) \land \text{walk}(x)]$.
   So, “a man” contains $\exists x[\text{man}(x) \land ?]$

b. every course is full. $\forall x[\text{course}(x) \rightarrow \text{full}(x)]$.
   So, “every course” contains $\forall x[\text{course}(x) \rightarrow ?]$

c. some professor taught. $\exists x[\text{professor}(x) \land \text{taught}(x)]$.
   So, “some professor” contains $\exists x[\text{professor}(x) \land ?]$

Now, what is the type for a quantified NP such as a man? Well, look at (a) above. If the predicate walk is typed $e \rightarrow t$, then we could say that this quantified NP is looking for a predicate to make a proposition: i.e., it’s looking for $e \rightarrow t$ to make a $t$. Hence, it’s type is $(e \rightarrow t) \rightarrow t$.

So, now let’s return to the examples above, and complete the semantic expression for each quantified NP. If we let $P$ and $Q$ stand for predicate variables, that is, variables of type $e \rightarrow t$, then we can finish the picture.

(2) a. a man: $\lambda P \exists x[\text{man}(x) \land P(x)]$

b. every course: $\lambda P \forall x[\text{course}(x) \rightarrow P(x)]$

c. some professor: $\lambda P \exists x[\text{professor}(x) \land P(x)]$

Now, let us imagine what the semantics of the quantifier all by itself is. Again, take away the predicative meaning of the noun that is the head of the NP, which is typed as $e \rightarrow t$. Then we can represent some, every, and the as follows. We are essentially following the definitions from Montague (1970).

(3) a. a: $\lambda P \lambda Q \exists x[P(x) \land Q(x)]$

b. every: $\lambda P \lambda Q \forall x[P(x) \rightarrow Q(x)]$

c. the: $\lambda P \lambda Q \exists x \forall y[[P(y) \leftrightarrow x = y] \land Q(x)]$

An example of the definite NP is shown below:
the teacher: $\lambda P \exists x \forall y [\text{teacher}(y) \leftrightarrow x = y] \land P(x)$

The type tree for a quantifier, such as every, combining with a common noun such as woman is illustrated below:

(5)

\[(e \rightarrow t) \rightarrow t\]
\[(e \rightarrow t) \rightarrow ((e \rightarrow t) \rightarrow t)\]
\[\text{every}\]
\[\text{woman}\]

The resulting functional expressions are called generalized quantifiers.

Now let us look at how quantifiers show up in transitive verbs, such as love, buy, and eat. First, consider the QNP in subject position, as with the sentence below.

(6) Every child loves Big Bird.

\[\forall y [\text{child}(y) \rightarrow \text{love}(y, \text{bb})] \land \forall y [\text{child}(y) \rightarrow \lambda x \{\text{love}(x, \text{bb})\}(y)]\]
\[
\lambda P \forall y [\text{child}(y) \rightarrow P(y)] (\lambda x [\text{love}(x, \text{bb})])
\]

2 Quantifier Substitution

It would be nice to think that compositional mechanisms do not change that much depending on the nature of the argument. But they do, and this is nowhere more apparent than when a quantified expression appears anywhere other than in subject position of the sentence, as illustrated below.

(7) a. Big Bird loves every child.

\[S \otimes\]
\[\lambda P \forall x [\text{child}(x) \rightarrow P(x)] \land \forall y [\text{child}(y) \rightarrow \text{love}(y, x)] \land \forall y [\text{child}(y) \rightarrow \lambda x [\text{love}(x, y, x)](y)]\]

To solve this kind of type mismatch, we're going to rework a very clever algorithm that Robin Cooper came up with, usually referred to as Cooper Storage (Cooper, 1975).

We will attempt to maintain the functional behavior for a predicate that we see in the syntax, as well as that which we saw above for constants in all argument positions. In other words, if a verb selects an argument that is typed as $e$, then it will take an argument of that type, even if the argument is a QNP such as every woman. I will refer to this as a Quantifier Substitution (QS), and informally it looks like the following.
(8)  a. Big Bird loves every child. \( \implies \)
    b. Big Bird loves C. \( \{C / \text{every child}\}_\sigma \).

This says that there is a substitution, \( \sigma \), where the semantics for “every child” is replace by the constant, \( C \). Notice that it is very similar to the quantifier elimination rules in Chapter 2 of the Semantics book, where we can infer an arbitrary constant, \( a \), in place of a quantifier scoping over a formula. Then, once the function application semantics of the entire expression has been worked through, we substitute the two expressions, which then results in further function application possibilities.

(9) Quantifier Substitution:
    For every expression, \( \gamma \), in a sentence, we associate a body, \( \alpha \), and the set of quantifier substitutions, \( \Sigma \), where
    \[
    \gamma = \alpha \{ \Sigma \}
    \]

We now define a rule called Substitution Application. This applies to each substitution, \( \sigma_i \) in \( \Sigma \), and it performs the following operation:

(10) Substitution Application:
    \[
    \alpha \{ \sigma_u \} \implies \sigma_u(\lambda u \alpha[u])
    \]

Let’s give a simple example to see how this works. We return to the derivation that broke earlier, namely, “Big Bird loves every child”.

(11) STEP-BY-STEP:
    a. Big Bird loves every child.
    b. loves: \( \lambda x \lambda y [\text{love}(y, x)] \)
    c. every child: \( \lambda P \forall x [\text{child}(x) \to P(x)] \)
    d. Quantifier Substitution (QS): \( C : e, [C / \lambda P \forall x [\text{child}(x) \to P(x)]]_\sigma \)
    e. Function Application: \( \lambda x \lambda y [\text{love}(y, x)] : e \to (e \to t), C : e \implies \lambda y[\text{love}(y, C)] : e \to t \)
    f. VP now denotes: \( \lambda y[\text{love}(y, C)] \{ \sigma \} \)
    g. Function Application: \( \lambda y[\text{love}(y, C)] \{ \sigma \}(bb) \implies [\text{love}(bb, C)] \{ \sigma \} \)
    h. Substitution Application: \( [\text{love}(bb, C)] \{ \sigma \} \implies \lambda P \forall x [\text{child}(x) \to P(x)][\lambda y[\text{love}(bb, y)]] \)
    i. Function Application: \( \lambda P \forall x [\text{child}(x) \to P(x)][\lambda y[\text{love}(bb, y)]] \implies \forall x[\text{child}(x) \to \lambda y[\text{love}(bb, y)](x)] \)
    j. Function Application: \( \forall x[\text{child}(x) \to \lambda y[\text{love}(bb, y)](x)] \implies \forall x[\text{child}(x) \to \text{love}(bb, x)] \)
    k. \( \odot \)
(12) TREE DERIVATION:
   a. Big Bird loves every child.
   b. $\forall \text{child}(x) \rightarrow \text{love}(bb, x)]$ $\otimes$
      $\forall \lambda P \forall \text{child}(x) \rightarrow P(x)]((\lambda y[\text{love}(bb, y)])$
      $\forall \text{love}(bb, C)\{[C/\lambda P \forall \text{child}(x) \rightarrow P(x)]_{\sigma}\}$

Big Bird: $e$, bb  VP: $e \rightarrow t$, $\lambda x[\text{love}(x, C)]\{[C/\lambda P \forall \text{child}(x) \rightarrow P(x)]_{\sigma}\}$

Big Bird  $\forall e \rightarrow (e \rightarrow t)$  NP: $(e \rightarrow t) \rightarrow t$, $\lambda P \forall \text{child}(x) \rightarrow P(x)$

If the quantifier substitution operation is similar to quantifier elimination, then the substitution application is similar to a successful arrow introduction, illustrating that the type shifting of the QNP to an individual was successful.

This technique also works for embedded quantifiers and coordinate NP constructions, which we will get to shortly. We will get to this topic in the next section of the handout.

3 Quantifier Embedding

In this section, we examine how a quantifier embedded within another quantified expression can be interpreted. We learn that substitutions can take different scopes over the expressions

This technique also works for embedded quantifiers and coordinate NP constructions, which we will get to shortly. Now consider a QNP embedded within another QNP, such as that shown below.

(13) John bought a picture of every student.

Assume that there are two readings: (14a) with wide-scope on every student; and (14b) with wide-scope on a picture.

(14) a. $\forall \text{student}(x) \rightarrow \exists y[\text{picture}(y) \wedge of(y, x) \wedge buy(j, y)]$
   b. $\exists y[\text{picture}(y) \wedge \forall \text{student}(x) \rightarrow of(y, x) \wedge buy(j, y)]$

Most speakers think that the first interpretation above (14a) is much more natural, and that the “single picture” reading is sort of hard get, without additional work or context, as shown below:

(15) a. John bought a picture of every student together.
   b. John bought a picture of all the students.

In any case, let’s go ahead and do the substitutions for the two quantifiers. First, the one that is embedded.

(16) a. (“every student”, $(e \rightarrow t) \rightarrow t, \lambda P \forall \text{student}(x) \rightarrow P(x)]$
    b. QS: (“every student”, $e, C_1), [C_1/\lambda P \forall \text{student}(x) \rightarrow P(x)]_{\sigma_1}$
Now let us derive the interpretation for the complex QNP containing $C_1$ under substitution $\sigma_1$.

(17) a. (“picture”, $e \to (e \to t)$, $\lambda x \lambda y[picture(y) \land of(y, x)]$

b. (“picture of $C_1$”, $e \to t$, $\lambda y[picture(y) \land of(y, C_1)]\{\sigma_1\}$

c. (“a picture of $C_1$”, $(e \to t) \to t$, $\lambda P \exists x[picture(x) \land of(x, C_1) \land P(x)]\{\sigma_1\}$

Notice where the substitution $\sigma_1$ ended up in (17c). Up till now, we haven’t really had to worry about the scope of the substitution itself: that is, how big of an expression does a substitution $\sigma$ attach to, anyway? According to our rule of Quantifier Substitution, repeated in (18),

(18) Quantifier Substitution:
For every expression, $\gamma$, in a sentence, we associate a body, $\alpha$, and the set of quantifier substitutions, $\Sigma$,

$$\gamma = \alpha\{\Sigma\}$$

$\sigma$ attaches to the entire expression, but it seems as though it can take a narrow attachment (or scope) as well; namely, attaching to the literal that contains the constant, $C_i$; that is, $\sigma_i$ can attach to either the expression within which the substitution was made, $\alpha$, or to a smaller literal within this expression; namely (19).

(19) (“picture of $C_1$”, $e \to t$, $\lambda y[picture(y) \land of(y, C_1)]\{\sigma_1\}$)

This in turn would give a “narrower scope” to the substitution when combined with the quantifier. So, rather than (17c), we derive (20).

(20) (“a picture of $C_1$”, $(e \to t) \to t$, $\lambda P \exists x[picture(x) \land of(x, C_1)]\{\sigma_1\} \land P(x)$)

Let’s derive the wide-scope reading for every student first.

(21) STEP-BY-STEP:

a. John bought a picture of every student.
b. buy: $\lambda x \lambda y[buy(y, x)]$
c. picture: $\lambda x \lambda y[picture(y) \land of(y, x)]$
d. every student: $\lambda P \forall x[student(x) \to P(x)]$
e. Quantifier Substitution (QS): $C_1: e, [C_1/\lambda P \forall x[student(x) \to P(x)]]\{\sigma_1\}$
f. Function Application: $\lambda x \lambda y[picture(y) \land of(y, x) : e \to (e \to t)], C_1 : e \implies \lambda y[picture(y) \land of(y, C_1)]\{\sigma_1\} : e \to t$
g. a picture: $\lambda P \exists y[picture(y) \land of(y, C_1) \land P(x)]\{\sigma_1\}$
h. Quantifier Substitution (QS): $C_2: e, [C_2/\lambda P \exists y[picture(y) \land of(y, C_1) \land P(x)]]\{\sigma_1\}\{\sigma_2\}$
i. Function Application: $\lambda x \lambda y[buy(y, x)] : e \to (e \to t)], C_2 : e \implies \lambda y[buy(y, C_2)] : e \to t$
j. Function Application: $\lambda y[buy(y, C_2)] : e \to t], j : e \implies buy(j, C_2)\{\sigma_2\}$
k. Substitution Application: $\lambda P \exists y[picture(y) \land of(y, C_1) \land P(x)]\{\sigma_1\}(\lambda z[buy(j, z)]) \implies \exists y[picture(y) \land of(y, C_1) \land buy(j, y)]\{\sigma_1\}$
m. Substitution Application: $\lambda P \forall x[student(x) \to P(x)](\lambda w \exists y[picture(y) \land of(y, w) \land buy(j, y)])$
n. Function Application: $\forall x[student(x) \to \exists y[picture(y) \land of(y, x) \land buy(j, y)]]$
o. $\odot$
Now let us derive the narrow-scope reading for every student.

(22) STEP-BY-STEP:
   a. John bought a picture of every student.
   b. buy: $\lambda x \lambda y [\text{buy}(y, x)]$
   c. picture: $\lambda x \lambda y [\text{picture}(y) \land \text{of}(y, x)]$
   d. every student: $\lambda P \forall x [\text{student}(x) \rightarrow P(x)]$
   e. Quantifier Substitution (QS): $C_1 : e, [C_1 / \lambda P \forall x [\text{student}(x) \rightarrow P(x)]]_{\sigma_1}$
   f. Function Application: $\lambda x \lambda y [\text{picture}(y) \land \text{of}(y, x) : e \rightarrow (e \rightarrow t), C_1 : e \implies$
   $\lambda y [\text{picture}(y) \land \text{of}(y, C_1) \{\sigma_1\}] : e \rightarrow t$
   g. a picture: $\lambda P \exists y [\text{picture}(y) \land \text{of}(y, C_1) \{\sigma_1\} \land P(x)]$
   h. Quantifier Substitution (QS): $C_2 : e, [C_2 / \lambda P \exists y [\text{picture}(y) \land \text{of}(y, C_1) \{\sigma_1\} \land P(x)]]_{\sigma_2}$
   i. Function Application: $\lambda x \lambda y [\text{buy}(y, x)] : e \rightarrow (e \rightarrow t), C_2 : e \implies$
   $\lambda y [\text{buy}(y, C_2)] : e \rightarrow t$
   j. Function Application: $\lambda y [\text{buy}(y, C_2)] : e \rightarrow t, j : e \implies$
   $\text{buy}(j, C_2) \{\sigma_2\}$
   k. Substitution Application: $\lambda P \exists y [\text{picture}(y) \land \text{of}(y, C_1) \{\sigma_1\} \land P(x)] (\lambda z [\text{buy}(j, z)]) \implies$
   l. $\exists y [\text{picture}(y) \land \lambda P \forall x [\text{student}(x) \rightarrow P(x)] (\lambda z [\text{of}(z, x)]) \land \text{buy}(j, y)]$
   m. Substitution Application:
   $\exists y [\text{picture}(y) \land \lambda P \forall x [\text{student}(x) \rightarrow P(x)] (\lambda z [\text{of}(z, x)]) \land \text{buy}(j, y)]$
   n. Function Application:
   $\exists y [\text{picture}(y) \land \forall x [\text{student}(x) \rightarrow \text{of}(y, x)] \land \text{buy}(j, y)]$
   o. $\Box$

To sum up, we can see that a quantified expression within another quantified expression has two options for recording the substitution:

(23) For every expression, $\gamma$, containing a body, $\alpha$, and a quantifier substitution, $[C_i / Q_i]_{\sigma_i}$, $\gamma$ can be encoded as either:
   a. $\alpha \{\sigma_i\}$
   b. $[\ldots \alpha_j \{\sigma_i\} \ldots]_{\alpha}$, where $\alpha_j$ contains $C_i$. 