1 The syntax of Predicate (First-Order) Logic

Besides keeping the connectives from Propositional Logic (PL), Predicate Logic (PrL) decomposes simple statements into smaller parts: predicates, terms and quantifiers.

(0) John is tall. \( T(j) \)
(1) John is taller than Bill. \( T(j, b) \)
(2) Everybody sleeps. \( \forall x [S(x)] \)
(3) Somebody likes David. \( \exists x [L(x, d)] \)

The syntax of PrL can be set forth like this.

Primitive vocabulary:

(4) Lexical entries, with a denotation of their own:
   a. A set of individual constants, represented with the letters \( a, b, c, d \ldots \)
   b. A set of individual variables \( x_0, x_1, x_2, \ldots y_0, y_1, y_2, \ldots \)
   c. A set of predicates, each with a fixed \( n \)-arity, represented by \( P, Q, R \ldots \)

(5) Symbols treated syncategorematically (introduced by a rule, not as a lexical entry):
   a. The PL logical connectives \( \neg, \lor, \land, \rightarrow \)
   b. The quantifier symbols \( \forall \) and \( \exists \)

(6) Syntactic rules:
   a. If \( P \) is a \( n \)-ary predicate and \( t_1 \ldots t_n \) are all terms, then \( P(t_1 \ldots t_n) \) is an atomic formula.
   b. If \( \phi \) is a formula, then \( \neg \phi \) is a formula.
   c. If \( \phi \) and \( \psi \) are formulae, then \( (\phi \land \psi), (\phi \lor \psi), (\phi \rightarrow \psi) \) are formulae too.
   d. If \( \phi \) is a formula and \( v \) is a variable, then \( \forall v \phi, \exists v \phi \) are formulae too.
   e. Nothing else is a formula in PrL.
Example of a formula:

\[ \exists x L(x, d) \]

Rule (6)d.

\[ L(x, d) \]

Rule (6)a.

\[ L \ x \ d \]

QUESTION 1: Draw the syntactic tree for the expressions in (8) that are well-formed formulae of PrL.

(8) a. \[ \exists(Qa \rightarrow PR(b)(c)) \]

b. \[ \forall x(P(x) \rightarrow \exists yQ(x, y)) \]

c. \[ \exists x_1 \forall x_2(P(x_1, x_2) \rightarrow (R(x_1)Q(x_2, a))) \]

Some syntactic notions:

(9) If \( x \) is any variable and \( \phi \) is a formula to which a quantifier has been attached by rule (6)d. to produce \( \forall x \phi \) or \( \exists x \phi \), then we say that \( \phi \) is the scope of the attached quantifier and that \( \phi \) or any part of \( \phi \) lies in the scope of that quantifier.

(10) An occurrence of a variable \( x \) is bound if it occurs in the scope of \( \forall x \) or \( \exists x \).

A variable is free if it is not bound.

(11) Formulae with no free variables are called closed formulae, formulae (simpliciter) or sentences.

Formulae containing a free variable are called open formulae or propositional functions.

2 Semantics of FOL

Recall the principle of compositionality – the meaning of a complex formula is determined by the meaning of its parts, plus the way those parts are combined.

(12) a. Lexicon: If \( \alpha \) is a constant or a predicate, then \( [\alpha]^w \) is specified by the interpretation function \( F \) (in the model/scenario \( w \)) that assigns set-theoretical objects to each constant/predicate (this is like saying that the semantic value of those constants/predicates is fixed like a definition or description in a dictionary or encyclopaedia).
Variables are different: their reference is not fixed by a dictionary or encyclopaedia (the way that reference of names and predicates is). They are a little bit like natural-language pronouns. So, we will define a separate function that just takes care of variables.

(12) b. Assignment function: If \( \alpha \) is a variable, then \([\alpha]^{w:g}\) is specified by a variable assignment function, which assigns individual entities in the universe of discourse \(D_e\) from our model \(w\) to each variable: \([\alpha]^{w:g} = g(\alpha)\)

The main difference between constants and variables is that the reference of the latter is variable (changeable).

(13) \(g^{d/v}\) reads as the variable assignment \(g'\) that is exactly like \(g\) except (maybe) for \(g(v)\), which equals the individual \(d\).

QUESTION 2: Complete the equivalences:

(14) \(g(x) = Mary\)

\(g^{Paul/x}(x) =\)

\(g^{Paul/Susan/x}(x) =\)

\(g^{Paul/Susan/y}(x) =\)

\(g(y) = Susan\)

\(g^{Paul/x}(y) =\)

\(g^{Paul/Susan/x}(y) =\)

\(g^{Paul/Susan/y}(y) =\)

Now that we know how to interpret lexical entries, we can write semantic rules for interpreting complex expressions. The rules below match the syntactic rules in (6) above:

(15) Interpreting atomic formulae:

a. If \(P\) is an \(n\)-ary predicate and \(t_1 t_n\) are all terms, then, for any scenario \(w\), \([P(t_1...t_n)]^{w:g} = 1\) iff \([t_1]^{w:g}, ... , [t_n]^{w:g} \in [P]^{w:g}\)

PL connectives are the same:

b. \([-\phi]^{w:g} = 1\) iff \([\phi]^{w:g} = 0\)

c. \([\phi \land \psi]^{w:g} = 1\) iff \([\phi]^{w:g} = 1\) and \([\psi]^{w:g} = 1\)

\([\phi \lor \psi]^{w:g} = 1\) iff \([\phi]^{w:g} = 1\) or \([\psi]^{w:g} = 1\) (or both)

\([\phi \rightarrow \psi]^{w:g} = 1\) iff \([\phi]^{w:g} = 0\) or \([\psi]^{w:g} = 1\)

And now, formulae with quantifiers:

d. If \(\phi\) is a formula and \(v\) is a variable, then, for any situation,

\([\forall v \phi]^{w:g} = 1\) iff \([\phi]^{w:g_{d/v}} = 1\) for all \(d \in D_e\)

\([\exists v \phi]^{w:g} = 1\) iff \([\phi]^{w:g_{d/v}} = 1\) for some \(d \in D_e\)

When can we drop the assignment superscript \(g\)? When it makes no difference to interpretation:

(16) For any formula \(\phi\), \([\phi]^{w} = 1\) iff, for all assignments \(g\), \([\phi]^{w:g} = 1\)

QUESTION 3: Let us take the situation \(s\) depicted in (17). Let us take a language PrL1 such that: the constants \(a, b,\) and \(c\) denote the individuals ■, •, and ♦, respectively, the unary predicate \(K\) denotes the set of individuals with a circle around them, and the binary predicate \(R\) denotes the relation encoded by the arrows.
Determine the truth value of the following formulae of PrL1 in s, justifying it in detail.

(17) a. \( \exists x \exists y \exists z \left( [R(x, y) \land K(y)] \land [R(x, z) \land \neg K(z)] \right) \)

b. \( \forall x (R(x, x)) \)

c. \( \forall x (R(x, x) \rightarrow \neg K(x)) \)

d. \( \exists x \exists y (R(x, y) \land [\neg K(x) \land \neg K(y)]) \)

**QUESTION 4:** Translate into PrL the following English sentences:

(18) a. John likes Susan.

b. John has a cat that he spoils.

c. Everything is bitter or sweet.

d. Either everything is bitter or everything is sweet.

e. There is something that everybody told Mary.

f. Everybody told Mary something.

g. If all logicians are smart, then Alfred is smart too.

h. Nobody came.

i. Nobody is loved by no one.

j. A whale is a mammal.

k. Barking dogs don’t bite.

l. Every student that bought a cat took it to the doctor.

m. Someone who promises something to somebody should do it.

3 Natural Deduction with FOL

Reasoning with PrL is very similar to reasoning with Propositional Logic, but we need some rules to deal with quantifiers. We begin with the elimination rule for \( \forall \), as it is the most intuitive one.

A bit of notation first: when we want to mention an open formula \( \phi \) which has at least one free occurrence of \( x \) in it, we can write this as \( \phi[x] \).

When we want to say instead that there is a mention of the individual constant \( c \) in \( \phi \), we can write this as \( \phi[c] \).

Also, when we want to mention a formula \( \phi \) in which every occurrence of a term \( u \) is replaced by term \( t \), we write this as \( \phi[t/u] \).
(19) (∀-elimination, or ∀E) If you have managed to write down ∀xφ, then you can go on and remove the quantifier, replacing every occurrence of x inside φ (that is, every x that was bound by ∀) with a constant (any constant).
1. ∀xφ we proved this somehow
2. φ[c/x] ∀E, line 1 - c is a constant

(20) Example of this rule in action:
Every man is mortal. Socrates is a man. Therefore, Socrates is mortal.
1. ∀x(man(x) → mortal(x)) basic assumption
2. man(s) basic assumption
3. man(s) → mortal(s) ∀E, line 1
4. mortal(s) → E, lines 2,3

In logic and philosophy, there is a big debate about this kind of reasoning, because it may (or may not) run into trouble in case if there is nothing at all in the universe. We’re going to ignore this case - notice, that in natural language, whenever we say “Everyone came to the party”, we assume that someone came to the party - that is, that our universe of discourse is not empty.

Next, let us consider the counterpart rule - ∀ Introduction.

(21) (∀-introduction, or ∀I) A universal quantifier can replace an individual constant (a name), but only if this name is not mentioned in any of the assumptions (basic, or additional undischarged ones).
1. φ[c] the constant c is not mentioned in basic assumptions Δ, nor in any additional undischarged assumptions we have made
2. ∀xφ[x/c] ∀I, line 1

(22) Example of this rule in action.
Every cat plays. Therefore, if the only things in the universe are cats (everything is a cat), everything in the universe plays.
1. ∀x(cat(x) → plays(x)) basic assumption
2. cat(s) → plays(s) ∀E, line 1
3. ∀x cat(x) additional assumption
4. cat(s) ∀E, line 3
5. plays(s) → E, lines 2,4
6. ∀x plays(x) ∀I, line 5, ok since no assumption mentions s
7. ∀x cat(x) → ∀x plays(x) → I lines 3-6

Now, let’s look at the existential quantifier. The introduction rule is very intuitive.

(23) (∃-introduction, or ∃I) If you have a statement about some constant, then you can deduce that this statement is true about something.
1. φ[c] we managed to write this down somehow...
2. ∃xφ[x/c] so we can conclude this, by rule ∃I, line 1

QUESTION 5. Use this rule, along with others we have so far, to show that

(24) ”Everyone came to the party” proves “Someone came to the party.”
Finally, the rule of $\exists$ elimination involves an additional assumption that must be discharged, by getting rid of the constant that replaces the existential quantifier, and replacing it back by an existential quantifier.

(25) \textbf{(\exists\text{-elimination, or } \exists E)} If you have an existential statement, you can replace the quantifier with a constant. This is valid, because we know there is some object about which the existential formula is true. However, we don’t know which specific object it is true about, so (i) we can’t choose a constant which has already been mentioned, and (ii) we must turn this constant back into an existential at the end.

1. $\exists x \phi[x]$
2. $\phi[c/x]$ so we go ahead and write this, by rule $\exists E$, line 1
3. $\psi[c]$ and we get almost where we need to be
4. $\exists x \psi[x/c]$ and the assumption is discharged, the name is done away with!

QUESTION 6. Use this rule, along with others we have so far, to show that

(26) "Every cat plays" and "There is a cat" proves "There is a cat that plays."