“What is that lambda thing?”
Formulating Abstractions with Higher Order Procedures
Structure and Interpretation of Computer Programs
Spring Term, 2015
Formulating Abstractions with Higher-Order Procedures

Idea: procedures identify common patterns

\[
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 (* 1 1))))))))
\]
\[
(* 10 (* 9 (* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 (* 1 1))))))))))
\]

==>

\[
(\text{define} \ (\text{factorial} \ n) \n  \quad (\text{if} \ (= n 0) \n      \quad 1 \n      \quad (* n (\text{factorial} (- n 1)))))
\]
How can we identify, abstract, and represent "higher order" common features?

\[
(+ 8 (+ 7 (+ 6 (+ 5 (+ 4 (+ 3 (+ 2 (+ 1 0)))))))) = \sum_{1 \leq k \leq 8} k
\]

\[
(+ 8^2 (+ 7^2 (+ 6^2 (+ 5^2 (+ 4^2 (+ 3^2 (+ 2^2 (+ 1^2 0)))))))) = \sum_{1 \leq k \leq 8} k^2
\]

\[
(+ (1/8^2) (+ (1/7^2) (+ (1/6^2) (+ (1/5^2) (+ (1/4^2) (+ (1/3^2) (+ (1/2^2) (+ (1/1^2 0)))))))))) = \sum_{0 \leq k \leq 8} 1/k^2
\]

All of these have the pattern

\[
\sum_{a \leq k \leq b} f(k)
\]
In Scheme, formal parameters of procedures can be bound to procedures as well as numbers, so that we can represent such common patterns:

```
(define (sum term a next b)
  (if (> a b)
      0
      (+ (term a)
          (sum term (next a) next b))))
```

Example of use:

```
(define (sum-squares a b)
  (sum square a 1+ b))
```

[NB: (define (1+ x) (+ 1 x)) ...
   ... but the unary 1+ is built into MIT Scheme]
(define (sum term a next b)
  (if (> a b)
      0
      (+ (term a)
          (sum term (next a) next b))))

(define (sum-squares a b) (sum square a 1+ b))

**Substitution model:**

(sum-squares 3 5)
(sum square 3 1+ 5)
(if (> 3 5) 0 (+ (square 3) (sum square (1+ 3) 1+ 5)))
(+ (square 3) (sum square (1+ 3) 1+ 5))
...
(+ 9 (sum square (1+ 3) 1+ 5))
(+ 9 (sum square 4 1+ 5))
...
(+ 9 (+ 16 (sum square 5 1+ 5)))
...
(+ 9 (+ 16 (+ 25 (sum square 6 1+ 5))))
...
(+ 9 (+ 16 (+ 25 (if (> 6 5) 0 (+ (square 6) (sum square (1+ 6) 1+ 5))))))
...
(+ 9 (+ 16 (+ 25 0)))
...
50
An infinite series for $\pi/8$:

$$\frac{1}{1\times3} + \frac{1}{5\times7} + \frac{1}{9\times11} + \frac{1}{13\times15} + \ldots = \frac{\pi}{8}$$

Scheme code to sum the first $n$ terms of the series:

Note the (first) multiplier in the $n$-th term is $4n-3$

```scheme
(define (funny-square x) (* x (+ x 2)))

(define (term n)
  (/ 1 (funny-square (- (* 4 n) 3))))

(define (pi-sum n)
  (sum term 1 1+ n))
```
Definite integrals

(define (integral fn a b dx)
  (* dx
     (sum fn
         (+ a (/ dx 2))
         (lambda (x) (+ x dx))
         b)))

Question: what's this funny lambda thing? Alternative:

(define (integral fn a b dx)
  (define (increment x) (+ x dx))
  (* dx
     (sum fn
         (+ a (/ dx 2))
         increment
         b)))

Once you know what a procedure does, you can usually forget how

When we write (define (square x) (* x x)) we are specifying:
(1) a procedure that multiplies its argument by itself, and
(2) a name for that procedure (why not call it Bob, Chuck, or Dave?)

Alternatively:

(define square (lambda (x) (* x x)))

Then (lambda (x) (* x x)) evaluates to the (nameless) procedure, which we then name using define. In the code for integral, we give the procedure without giving a name to it.

Why would you name something? Because you use it more than once.
(define (integral fn a b dx)
  (* dx
    (sum fn
     (+ a (/ dx 2))
     (lambda (x) (+ x dx))
     b)))

Now use integral as a building block:

\[ \text{arctangent } a = \int_{0}^{a} \frac{dx}{1 + x^2} \]

(define (arctangent a)
  (integral (lambda (x) (/ 1 (1+ (square x))))
            0
            a
            .001))
A further, useful abstraction

(define (accumulate combiner null-value term a next b)
  (if (> a b)
      null-value
      (combiner (term a)
                (accumulate combiner null-value term (next a) next b))))

(define (sum term a next b)
  (accumulate + 0 term a next b))

(define (factorial n)
  (accumulate * 1 (lambda (x) x) 1 1+ n))

Expand the body of factorial using the substitution model (NB: a, b not substituted for):

(if (> a b)
  1
  (* ((lambda (x) x) a) (accumulate * 1 (lambda (x) x) (1+ a) 1+ b)))

What does the procedure (lambda (x) x) do?
Square roots, revisited.

A **fixed point** of a function \( f(-) \) is an input \( a \) such that \( f(a) = a \). To compute the square root of \( x \), find the fixed point of

\[
y + \frac{x}{y} \\
f(y) = \frac{y + \frac{x}{y}}{2}
\]

```scheme
(define (fixed-point fn guess)
  (if (good-enough? fn guess)
      guess
      (fixed-point fn (fn guess))))

(define (good-enough? fn guess)
  (< (abs (- (fn guess) guess)) .0001))

(define (sqrt x)
  (fixed-point (lambda (y) (average y (/ x y))) 1))
```
(define (fixed-point fn guess)
  (if (good-enough? fn guess)
      guess
      (fixed-point fn (fn guess))))

Note that procedure \texttt{fixed-point} does not always evaluate to something!

Substitution model:

(fixed-point 1+ 2)
(fixed-point 1+ 3)
...

Solve for $x$:

$8x + 3 = 15 + 6x$

Or: $x = (12 + 6x)/8$ [a recursive definition???]

(define (f x) (/ (+ 12 (* 6 x)) 8))

(fixed-point f 1)
;Value: 5.999996224522291

Is there an easier way to solve this equation? (You learned it in the 7th grade...)
Digression to crazy but true idea:  
*Recursion is about solving equations!*

Exercise:  solve the following equations for the red variable:

\[ 3x^2 - 5x + 4 = 6 \]

\[
\begin{align*}
\text{(define (g x) (sqrt (/ (+ 2 (* 5 x)) 3)))} \\
\text{(fixed-point g 1)} \\
\text{;Value: 1.999998974260502}
\end{align*}
\]

\[ x = 10 \]

What is \((\text{fixed-point (lambda (x) 10) 1)})?\]

\[ f(x) = f(-x) \]

\[ 3x^2 f''(x) + 2x f(x) - 10x^2(x) + 4x^3 = 0 \quad \text{(a differential equation...)}\]

\[ f = (\text{lambda (x) (* x x))} \]

\[ f = (\text{lambda (n) (if (= n 0) 1 (* n (f (- n 1)))))} \]

*(solution: the factorial function!)*
A fish weighs 20 pounds plus half its own weight---how much does the fish weigh?

\[ f = 20 + \frac{f}{2} \]
\[ = 20 + \frac{1}{2} (20 + \frac{f}{2}) \]
\[ = (1+ \frac{1}{2})20 + \frac{f}{4} \]
\[ = (1+ \frac{1}{2})20 + \frac{1}{4}(20 + \frac{f}{2}) \]
\[ = (1+ \frac{1}{2} + \frac{1}{4})20 + \frac{f}{8} \]
\[ ... = (1+ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...)20 + \frac{f}{...} \]

CS student (this was an exam question!): “The equation cannot be solved, because... the weight of the fish is defined in terms of itself” (!!!)

But we learn how to solve 1-variable linear equations in the 7th grade!

My guess: the student thought “20” was the base case, and “f/2” was the recursive call... And “f” was, of course, factorial!
Derivatives (from calculus, not finance...):
procedures as returned values

(define (deriv fn x dx)
  (/ (- (fn (+ x dx)) (fn x))
    dx))

This computes the derivative of a function at a point -- but what if what we want to compute is the derivative function?

(define (deriv fn dx)
  (lambda (x)
    (/ (- (fn (+ x dx)) (fn x))
      dx))
  ;Value deriv

This procedure returns another procedure as a value!

(define dsquare (deriv square .00001))
  ;Value: dsquare

(dsquare 2)
  ;Value: 4.000010000027032

(((deriv square .00001) 5)
  ;Value: 10.000009999444615
Derivatives and higher-order procedures:

```scheme
(define (deriv fn dx)
  (lambda (x)
    (/ (- (fn (+ x dx)) (fn x))
       dx))

;Value deriv

This procedure returns another procedure as a value!

(define dsquare (deriv square .00001))
;Value: dsquare

Substitution model:

(deriv square .00001)
(lambda (x) (/ (- (square (+ x .00001)) (square x)) .00001))

(dsquare 2)
(/ (- (square (+ 2 .00001)) (square 2)) .00001))
;Value: 4.000010000027032

((deriv square .00001) 5)
;Value: 10.000009999444615
```
Composing two functions:

(define (compose f g)
  (lambda (x) (f (g x))))
;Value: compose

((compose square 1+) 7)
;Value: 64

(define (repeated fn n)
  (if (= n 1)
      fn
      (compose fn (repeated fn (- n 1))))))
;Value: repeated

((repeated square 3) 5)
;Value: 390625
Composing two functions:

\[(\text{define } (\text{compose } f \ g) \ (\lambda (x) (f (g \ x)))\)]
;Value: compose

\[((\text{compose } \text{square} \ 1+) \ 7)\]
;Value: 64

Substitution model:

\[(((\lambda (x) (\text{square} \ (1+ \ x))) \ 7) \ (\text{square} \ (1+ \ 7)))\]
Substitution model:

(define (repeated fn n)
  (if (= n 1)
      fn
      (compose fn (repeated fn (- n 1))))))

((repeated square 3) 5)
  ((if (= 3 1)
       square
       (compose square (repeated square (- 3 1)))) 5)
  ...
  ((compose square (repeated square 2)) 5)
  ...
  ((compose square (compose square (repeated square 1))) 5)
  ...
  ((compose square (compose square square)) 5)
  ((compose square (lambda (x) (square (square (square x))))) 5)
  ((lambda (y) (square
     ((lambda (x) (square (square x))) y))) 5)
  (square (lambda (x) (square (square (square x)))) 5))
  (square (square (square 5)))
  ...
  390625

Notice how ((repeated square n) k) expands proportional to n.
A space-efficient alternative function composition:

```
(define (repeated fn n)
  (if (= n 1)
      fn
      (lambda (x) ((repeated fn (- n 1)) (fn x)))))
```

Notice that in the substitution model, we do not "evaluate under lambda"---that is, don't simplify the body of a lambda expression until the lambda expression is applied.

```
((repeated square 3) 5)
...
((lambda (x) ((repeated square (- 3 1)) (square x))) 5)
((repeated square (- 3 1)) (square 5))
((repeated square 2) 25)
...
((lambda (x) ((repeated square (- 2 1)) (square x))) 25)
((repeated square (- 2 1)) (square 25))
((repeated square 1) (square 25))
...
(square (square 25))
...
(square 625)
...
390625
```

The above process uses space constant in n, not linear in n as before.
Currying: single versus multiargument functions (a useful programming idiom...)

(+ 8 5)
;Value: 13

(define (plus n)
  (lambda (x) (+ n x)))
;Value: plus

(define 2+ (plus 2))
;Value: 2+

(2+ 5)
;Value: 7

((plus 8) 5)
;Value: 13

Haskell Curry (logician) [1900-1982]
There's a programming language named after this guy...

Analyzing functionality (data types, informally)

+: (Int * Int) --> Int
plus: Int --> (Int --> Int)
+ : (Int * Int) --> Int
plus: Int --> (Int --> Int)

(define (curry fn)
  (lambda (x)
    (lambda (y)
      (fn x y))))
; Value: curry

(define plus (curry +))
; Value: plus

Substitution model:

(((curry +) 8) 5)
(((lambda (x) (lambda (y) (+ x y))) 8) 5)
((lambda (y) (+ 8 y)) 5)
(+ 8 5)
13

Reverse "uncurrying"

(define (uncurry curried-fn)
  (lambda (x y) ((curried-fn x) y)))

Exercise: Check that + and (uncurry (curry +)) do the same thing...
Yet another crazy idea:
Programs are proofs, and types are theorems

\[ f : A \rightarrow B \]

(Or:) \( f \) is a proof of theorem \( A \rightarrow B \), in other words, a construction that turns any proof of \( A \) into a proof of \( B \)

\[ x : A \]

\[ (f x) : B \]

function application is \textit{modus ponens} from logic!
Higher order procedures and functionality --
What kind (type) of answers are supposed to be computed?

This is the key to solving the first problem set.
We write informal "types" to describe functionality.

type Height = Real
type Velocity = Real
type Fuel = Real
type Shipstate = Height * Velocity * Fuel \[\text{[a 3-field record...]}\]
type Fuel-burnrate = Real
type Strategy = Shipstate --> Fuel-burnrate

Problem 3:

random-choice: (Strategy * Strategy) --> Strategy

Note
(Strategy * Strategy) --> Strategy =
((Shipstate --> Fuel-burnrate) * (Shipstate --> Fuel-burnrate)) -->
(Shipstate --> Fuel-burnrate)

...a real higher-order procedure!...
Question: what kind of "random choice" is this? Once, or continually?
type Height = Real
type Velocity = Real
type Fuel = Real

**Problem 4:**

**height-choice:** (Strategy * Strategy * Int) --> Strategy

**Problem 5:**

**choice:** (Strategy * Strategy * (Shipstate --> Boolean)) --> Strategy

Q: Can we apply this type of analysis to politics and public policy?
Procedures are **first-class** values:
(a concept introduced by Christopher Strachey)

They can be named by variables.
They can be parameters of procedures.
They can be returned as results of procedures.

What else is first-class in Scheme?
Integers? **define**?
Some useful syntactic sugar: `let`

\[ f(x,y) = (x^2 + y^2)(x^2 - y^2) \]

\[
\begin{align*}
(\text{define} & \ (\text{fn} \ x \ y) \\
& \ (* \ (+ \ (\text{square} \ x) \ (\text{square} \ y)) \\
& \ (- \ (\text{square} \ x) \ (\text{square} \ y)))) \\
\end{align*}
\]

Better: (or, why is it better? or is it?)

\[
\begin{align*}
(\text{define} & \ (\text{fn} \ x \ y) \\
& \ (\text{define} \ (\text{helper} \ a \ b) \ (* \ (+ \ a \ b) \ (- \ a \ b)) \\
& \ (\text{helper} \ (\text{square} \ x) \ (\text{square} \ y))) \\
\end{align*}
\]

Why name something used only once?

\[
\begin{align*}
(\text{define} & \ (\text{fn} \ x \ y) \\
& \ ((\lambda \ (a \ b) \ (* \ (+ \ a \ b) \ (- \ a \ b))) \ (\text{square} \ x) \ (\text{square} \ y))) \\
\end{align*}
\]

Hard to read, so instead use this syntactic sugar:

\[
\begin{align*}
(\text{define} & \ (\text{fn} \ x \ y) \\
& \ (\text{let} \ ((a \ (\text{square} \ x)) \\
& \ b \ (\text{square} \ y))) \\
& \ (* \ (+ \ a \ b) \ (- \ a \ b))) \\
\end{align*}
\]
(let ((v1 e1) (v2 e2) ... (vn en)) b) ==> ((lambda (v1 v2 ... vn) b) e1 e2 ... en)

Pitfall: Dependence of variable definitions...
(Variables in above are "bound" simultaneously, not sequentially, so they cannot depend on each other.)

(define x 2)
;Value: x

(let ((x 3) (y (+ x 2))) (* x y))
;Value: 12

Rewrite the last expression as:

((lambda (x y) (* x y)) 3 (+ x 2))

where does the red x get its value?
It is a "free" (not bound) variable

An analogy: how much is 2 and 2?
Math nerd: “4”
Engineer: “3.99998”
Lawyer: “How much would you like it to be?
[Programming language design is like being the lawyer... what do we want the meaning to be?]}

Using the substitution model, rewrite the last expression as:

(((lambda (x) (let ((y (+ x 2))) (* x y))) 3)
(((lambda (x) ((lambda (y) (* x y)) (+ x 2))) 3)
(((lambda (y) (* 3 y)) (+ 3 2)))
(((lambda (y) (* 3 y)) 5)
(* 3 5)
15