Reasoning about Knowledge and Continuity

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Abstract

The aim of this paper is to extend the modal logic of knowledge due to Moss and Parikh by state transformers arising, eg, from actions of agents. The peculiarity of Moss and Parikh's approach lies in the fact that topological reasoning is supported. Emphasizing this we are mainly concerned with the idea of continuity here, which can be captured with the aid of the extended framework. We define and discuss an appropriate language, and we study the accompanying logical system. It turns out that certain expressive means from a particular extension of the commonly employed modal formalism, viz hybrid logic, are very useful for our purposes. With that, we obtain the finite axiomatizability, soundness and completeness, and decidability, of the new logic.

Keywords: Reasoning about knowledge, topological reasoning, modal logic, continuous functions, hybrid logic

Introduction

We take up Moss and Parikh's approach to reasoning about knowledge of some agent; cf Moss & Parikh 1992 and, more detailedly, Dabrowski, Moss, & Parikh 1996. The bi-modal system for knowledge and *effort* released in these papers facilitates a fairly abstract description of the process of gaining knowledge. As a bonus, a certain *topological* component of knowledge is revealed. In fact, since knowledge is represented by the space of all *knowledge states* of the agent, knowledge acquisition appears as a *shrinking procedure* or *approximation* regarding this space of sets. Thus, certain notions from topology like *closeness* or *neighbourhood* enter the realm of knowledge in a natural way.

Moss and Parikh called their system topologic. We recall some features of the language of topologic a little more detailedly. As it has just been indicated, formulas may contain two one-place operators: a modality K describing knowledge and another one, $\Box$, describing effort. The domains for evaluating formulas are *set spaces* $(X, \mathcal{O}, V)$ consisting of a non-empty set X of states, a set $\mathcal{O}$ of subsets of X representing the knowledge states of the agent, and a valuation V determining the states where the atomic propositions are true. The operator K then quantifies across any knowledge state $U \in \mathcal{O}$, whereas $\Box$ quantifies 'downward' across $\mathcal{O}$. That is, more knowledge, i.e., closer proximity to states of 'complete' knowledge, can be achieved by descending with respect to the set inclusion relation inside $\mathcal{O}$, and just this is modeled by $\Box$.

Several classes of set spaces have been investigated from this topological view of knowledge, among these the ordinary topological ones; cf Georgatos 1994 and Dabrowski, Moss, & Parikh 1996. However, more expressive power is needed to capture more special notions from topology like separation or connectedness.

To this end, a *sorted,* i.e., both state- and set-sensitive *hybrid version* of topologic was introduced in Heinemann 2003, and further developed in Heinemann 2004 and Heinemann 2005a. The language considered in these papers provides the starting point to the following, where the concept of *continuity* is studied in the context of set spaces. Thus we pick up an old question from Moss & Parikh 1992 here, which was raised in connection with the knowledge-theoretic analysis of recursive function theory; see Moss & Parikh 1992, Sec. 6.

Over and above that, continuity is important for *computability on non-discrete structures*. In fact, in case of real-valued functions, for instance, we have that continuity is a necessary condition for computability. Thus proving the non-effectiveness of an operation *op* involving the reals can be done by showing the non-continuity of *op*. (This is the most common method for that, actually; see Weihrauch 2000 for the details.)

The relevance of continuity to reasoning about knowledge can basically be gathered from the definition already, assuring a state transformation f of its compatibility with the knowledge states of the agent. This applies, in particular, to the case where f represents some *action* of the agent; cf Fagin *et al.* 1995, Sec. 5.1.

In order to deal with continuity properly we must first of all be able to speak about functions in the logical language. This could be done either implicitly by using a respective modality, or explicitly by representing functions through terms, i.e., treating these like functions in first-order logic. Following the second approach would lead to 'algebraizing hybrid logic'; cf Tzanis 2005. However, the first alternative turns out to be the right one for our purposes. This comes as a certain surprise in view of the semantics of topologic.

The content of this paper is as follows. In the next section we define precisely our hybrid language for (continuous) functions on set spaces, and we give a couple of examples concerning expressive power. Afterwards, we axiomatize the arising hybrid logic of set spaces with functions, and we prove a corresponding soundness and completeness theorem. We then deal with the problem of deciding the new logic. Finally, we give a brief summary and point to future research.

All that we need from hybrid logic is contained in the textbook Blackburn, de Rijke, & Venema 2001, Sec. 7.3, which is also taken as a reference to the facts from modal logic we use in this paper.

Besides the results from Heinemann 2005b quoted in the proofs below, the present paper is, taken by and large, self-contained.

We conclude this introduction with some general remarks on related approaches to the connection between modal logic and topology. (Concerning the connection between knowledge and action, two of the existing approaches are referred to in the final section of this paper.) The topological interpretation of modal logic dates from the work of McKinsey, cf McKinsey 1941, and has been revitalized a couple of years ago; cf, eg, Aiello, van Benthem, & Bezhanishvili 2003. Meanwhile, the field split up into diverse branches of research. The forthcoming textbook Aiello, Pratt-Hartmann, & van Benthem (Eds.) 2006 gives an overview of the state of the art. The present paper goes best with the chapter on topology and epistemic logic there.

Defining the language

We add to the language of topologic two sets of nominals, the global modality (cf Blackburn, de Rijke, & Venema, Sec. 7.1), and a set of function symbols. The denotation of a nominal is intended to be either a unique state or a distinguished set of states, and function symbols may be applied to arbitrary formulas.

Let $\mathrm{PROP} = \{p,q,\ldots\}$, $\mathrm{N}_{stat} = \{i,j,\ldots\}$ and $\mathrm{N}_{sets} = \{A,B,\ldots\}$ be three mutually disjoint denumerable sets of symbols called *proposition variables, names of states* and *names of sets,* respectively. Moreover, let $\mathscr{F} = \{f,g,\ldots\}$ be a set of one-place function symbols. Then, the set WFF of *well-formed formulas* over $\mathrm{PROP}\cup\mathrm{N}_{stat}\cup\mathrm{N}_{sets}\cup\mathscr{F}$ is defined by the rule

$$\alpha ::= p \mid i \mid \neg\alpha \mid \alpha\wedge\beta \mid K\alpha \mid \Box\alpha \mid \mathsf{A}\alpha \mid [f]\alpha.$$

The missing boolean connectives are treated as abbreviations, as needed. The duals of the modal operators K, $\Box$, A and $[f]$ are denoted L, $\Diamond$, E and $\langle f\rangle$, respectively.

We now turn to semantics. For a start, we define the relevant domains. We let $\mathscr{P}(X)$ designate the powerset of a given set X.

Definition 1 (Set frames and spaces) *1. A* set frame with functions *is a triple*

$$\mathscr{S} := \left(X,\mathscr{O},\{F_f \mid f\in\mathscr{F}\}\right),$$

where X is a non-empty set, $\mathscr{O}\subseteq\mathscr{P}(X)$ a set of subsets of X such that $\{X,\emptyset\}\subseteq\mathscr{O}$, and, for every $f\in\mathscr{F}$,

$$F_f : X \longrightarrow X$$

a (total) function.[1]

2. Let $\mathscr{S} := \left(X,\mathscr{O},\{F_f \mid f\in\mathscr{F}\}\right)$ be a set frame with functions. The set

$$\mathscr{N}_{\mathscr{S}} := \{(x,U) \mid x\in U \text{ and } U\in\mathscr{O}\}$$

is called the set of neighbourhood situations *of $\mathscr{S}$.*

*3. Let $\mathscr{S} = \left(X,\mathscr{O},\{F_f \mid f\in\mathscr{F}\}\right)$ be a set frame with functions. An $\mathscr{S}$–*valuation *is a mapping*

$$V : \mathrm{PROP}\cup\mathrm{N}_{stat}\cup\mathrm{N}_{sets} \longrightarrow \mathscr{P}(X)$$

such that

(a) $V(i)$ is either $\emptyset$ or a singleton subset of X for every $i\in\mathrm{N}_{stat}$, and

(b) $V(A)\in\mathscr{O}$ for every $A\in\mathrm{N}_{sets}$.

4. Let $\mathscr{S} = \left(X,\mathscr{O},\{F_f \mid f\in\mathscr{F}\}\right)$ be a set frame with functions and V an $\mathscr{S}$–valuation. Then,

$$\mathscr{M} := \left(X,\mathscr{O},\{F_f \mid f\in\mathscr{F}\},V\right)$$

is called a set space with functions *(or, in short, an* SSF*). We then say that $\mathscr{M}$ is* based on *$\mathscr{S}$.*

Note that the definition takes into account that nominals may have an empty denotation. This is appropriate for the purposes of this paper, but not usual for standard hybrid logic.

Now, let an SSF $\mathscr{M}$ be given. We define the relation of satisfaction, $\models_{\mathscr{M}}$, between neighbourhood situations of the underlying frame and formulas in WFF. In the following, neighbourhood situations are written without brackets.

Definition 2 (Satisfaction and validity) *Let*

$$\mathscr{M} := \left(X,\mathscr{O},\{F_f \mid f\in\mathscr{F}\},V\right)$$

be an SSF based on $\mathscr{S} = \left(X,\mathscr{O},\{F_f \mid f\in\mathscr{F}\}\right)$, and let $x,U\in\mathscr{N}_{\mathscr{S}}$ be a neighbourhood situation of $\mathscr{S}$. Then

$$\begin{array}{lcl}
x,U\models_{\mathscr{M}} p & :\Longleftrightarrow & x\in V(p)\\
x,U\models_{\mathscr{M}} i & :\Longleftrightarrow & x\in V(i)\\
x,U\models_{\mathscr{M}} A & :\Longleftrightarrow & V(A)=U\\
x,U\models_{\mathscr{M}} \neg\alpha & :\Longleftrightarrow & x,U\not\models_{\mathscr{M}}\alpha\\
x,U\models_{\mathscr{M}} \alpha\wedge\beta & :\Longleftrightarrow & x,U\models_{\mathscr{M}}\alpha \text{ and } x,U\models_{\mathscr{M}}\beta\\
x,U\models_{\mathscr{M}} K\alpha & :\Longleftrightarrow & \text{for all } y\in U : y,U\models_{\mathscr{M}}\alpha\\
x,U\models_{\mathscr{M}} \Box\alpha & :\Longleftrightarrow & \left\{\begin{array}{l}\text{for all } U'\in\mathscr{O} : \text{if } x\in U'\\ \subseteq U,\ \text{then } x,U'\models_{\mathscr{M}}\alpha\end{array}\right.\\
x,U\models_{\mathscr{M}} \mathsf{A}\alpha & :\Longleftrightarrow & \left\{\begin{array}{l}\text{for all } y,U'\in\mathscr{N}_{\mathscr{S}} :\\ y,U'\models_{\mathscr{M}}\alpha\end{array}\right.\\
x,U\models_{\mathscr{M}} [f]\alpha & :\Longleftrightarrow & \left\{\begin{array}{l}\text{for all } y,U'\in\mathscr{N}_{\mathscr{S}} : \text{if}\\ F_f(x)=y,\ \text{then}\\ y,U'\models_{\mathscr{M}}\alpha,\end{array}\right.
\end{array}$$

where $p\in\mathrm{PROP}$, $i\in\mathrm{N}_{stat}$, $A\in\mathrm{N}_{sets}$, $f\in\mathscr{F}$, and $\alpha,\beta\in$ WFF. *In case $x,U\models_{\mathscr{M}}\alpha$ is true we say that α* holds in *$\mathscr{M}$* at *the neighbourhood situation x,U. Furthermore, a formula α is called* valid in *$\mathscr{M}$ iff it holds in $\mathscr{M}$ at every neighbourhood situation of $\mathscr{S}$. (Manner of writing: $\mathscr{M}\models\alpha$.)*

[1] We also consider set frames without functions, i.e., pairs $(X,\mathscr{O})$, later on.

Note that the formulas of the form $i \wedge A$, where $i \in \mathrm{N}_{stat}$ and $A \in \mathrm{N}_{sets}$, can be taken as names of neighbourhood situations. The hybrid satisfaction operator associated with such a name then reads $\mathsf{E}(i \wedge A \wedge \ldots)$. Thus these formulas act like 'proper' nominals in SSFs.

In the remaining part of this section, two examples are given which show the appropriateness of the just defined language.

Example 3 *The set frame considered in the following provides the 'generic' examples for the languages extending the one underlying* topologic. *Take the set $\mathscr{C}$ of all infinite* 0–1–*sequences. (The letter '$\mathscr{C}$' should remind one of the* Cantor Space.*) This set is endowed with a natural topology, $\mathscr{T}$, which is often called the* initial-segment topology *since a basis, $\mathscr{B}$, of $\mathscr{T}$ is determined by the set of all finite initial segments of elements of $\mathscr{C}$ (in a way suggesting itself). Let $\mathscr{S} := (\mathscr{C}, \mathscr{B} \cup \{\emptyset\})$. Obviously, $\mathscr{S}$ is a set frame. Note that $\mathscr{C}$ can be depicted as the full infinite binary tree; moreover, every node of this tree can be annotated with the set $U \in \mathscr{B}$ which is determined by the initial segment leading to this node. Thus a* tree of subsets *of $\mathscr{C}$ results. The frame $\mathscr{S}$ is basically the domain by means of which the procedures computing binary streams can be modeled.*

Now, suppose that some procedure P enumerating certain real numbers $\rho_1, \rho_2, \rho_3, \ldots$ is given. (It is assumed here that the computation of every single real number ρ is realized via a suitable representation, *which may, eg, come from the intuition that the output of P encodes a fast-converging Cauchy sequence having limit ρ; cf Weihrauch 2000.) Suppose additionally that we would like to test whether a sufficiently good approximation to, say, the real number π is enumerated. Since we eventually know if this is* not *the case with the actually computed number (due to our model of computation), we then can restart the enumeration procedure with a possibly better trial.*

The new language makes it possible to specify a corresponding 'protocol'. For that, let restart *be a function symbol and π a nominal having the intended meaning. The desired 'process code' then reads*

$$K\neg\pi \rightarrow \langle \mathsf{restart} \rangle \top.$$

The premise of this implication should be taken as an await*-statement. Note that this process is really 'enabled' in case the actual computation (having output, say, σ) is not the right one. In fact, the formula*

$$\Diamond K \neg\pi$$

then holds at every neighbourhood situation of the form σ, U of $\mathscr{S}$.

The next example shows that the new language is really expressive enough to capture continuity. For the sake of simplicity, we confine ourselves to a single function.

Definition 4 (Continuous functions) *Let $\mathscr{F}$ consist of a single function symbol f. Let $\mathscr{S} = (X, \mathscr{O}, F_f)$ be a corresponding frame. Then, F_f is called* continuous (with respect to $\mathscr{O}$) *iff for all $x \in X$ and $U, \tilde{U} \in \mathscr{O}$ such that $x \in U$ and $F_f(x) \in \tilde{U}$ there exists $U' \in \mathscr{O}$ satisfying $x \in U' \subseteq U$ and $F_f(U') \subseteq \tilde{U}$.*

Note that this definition meets continuity in the topological sense in case $\mathscr{O}$ is a system of neighbourhoods of every point with respect to a given topology, actually.

Proposition 5 *Let $\mathscr{S} = (X, \mathscr{O}, F_f)$ be as above. Then, F_f is continuous iff*

$$\mathscr{M} \models \langle f \rangle A \rightarrow \Diamond K \langle f \rangle A$$

holds for all $A \in \mathrm{N}_{sets}$ and SSFs $\mathscr{M}$ based on $\mathscr{S}$.

The proof of Proposition 5 can be done in a standard manner and is, therefore, omitted.

The logic

In this section, we first introduce an axiomatization of the hybrid logic of set spaces with functions. We then deal with the question of completeness. Finally, we take continuity into account, too.

To begin with, we list the usual axioms of topologic from Dabrowski, Moss, & Parikh 1996:

1. All instances of tautologies.
2. $K(\alpha \rightarrow \beta) \rightarrow (K\alpha \rightarrow K\beta)$
3. $K\alpha \rightarrow \alpha \wedge KK\alpha$
4. $L\alpha \rightarrow KL\alpha$
5. $(p \rightarrow \Box p) \wedge (\Diamond p \rightarrow p)$
6. $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$
7. $\Box\alpha \rightarrow \alpha \wedge \Box\Box\alpha$
8. $K\Box\alpha \rightarrow \Box K\alpha$,

where $p \in \mathrm{PROP}$ and $\alpha, \beta \in \mathrm{WFF}$. The final schema of this group, usually called the *Cross Axiom,* cf Dabrowski, Moss, & Parikh 1996, is characteristic of every logic of knowledge and effort.

The next group of axioms concerns names:

9. $(i \rightarrow \Box i) \wedge (\Diamond i \rightarrow i)$
10. $i \wedge \alpha \rightarrow K(i \rightarrow \alpha)$
11. $A \rightarrow KA$
12. $\mathsf{A}(A \wedge \alpha \rightarrow L\beta) \vee \mathsf{A}(A \wedge \beta \rightarrow L\alpha)$,

where $i \in \mathrm{N}_{stat}$, $A \in \mathrm{N}_{sets}$ and $\alpha, \beta \in \mathrm{WFF}$. The formulas of this group provide for the right nominal structure of the canonical model.

The following three axioms are responsible for the fact that really a structure of set space can be ensured with the aid of that model.

13. $i \wedge A \wedge \mathsf{E}(j \wedge B) \rightarrow \mathsf{E}(\Diamond(i \wedge A) \wedge L\Diamond(j \wedge B))$
14. $i \wedge A \rightarrow \Box(\Diamond(i \wedge A) \rightarrow i \wedge A)$
15. $K(\Diamond B \rightarrow \Diamond A) \wedge L\Diamond B \rightarrow \Box(A \rightarrow L\Diamond B)$,

where $i, j \in \mathrm{N}_{stat}$ and $A, B \in \mathrm{N}_{sets}$.

The global modality is now axiomatized as usual; cf Blackburn, de Rijke, & Venema 2001, Sec. 7.1.

16. $\mathsf{A}(\alpha \rightarrow \beta) \rightarrow (\mathsf{A}\alpha \rightarrow \mathsf{A}\beta)$
17. $\mathsf{A}\alpha \rightarrow \alpha \wedge \mathsf{AA}\alpha$
18. $\alpha \rightarrow \mathsf{AE}\alpha$
19. $\mathsf{A}\alpha \rightarrow K\Box\alpha \wedge [f]\alpha$,

where $\alpha, \beta \in$ WFF and $f \in \mathscr{F}$.

Finally, for every operator $[f]$ the functionality of F_f has to be captured. This is done in the following way:

20. $[f](\alpha \to \beta) \to ([f]\alpha \to [f]\beta)$
21. $[f]\alpha \to \langle f \rangle \alpha$
22. $\langle f \rangle i \to [f]i$
23. $\Diamond \langle f \rangle i \to \Box \langle f \rangle i$
24. $\langle f \rangle i \wedge \mathsf{E}(i \wedge A) \to \langle f \rangle (i \wedge A)$,

where $f \in \mathscr{F}$, $i \in \mathrm{N}_{stat}$, $A \in \mathrm{N}_{sets}$ and $\alpha, \beta \in$ WFF.

Note that, given a set frame with functions

$$\mathscr{S} = \left(X, \mathscr{O}, \{F_f \mid f \in \mathscr{F}\}\right),$$

every F_f transforms *states* rather than the semantic atoms of our language, i.e., neighbourhood situations. Thus, only Axiom 21, i.e., *seriality,* remains from the usual modal axiomatization of functionality (besides Axiom 20), while *determinism* has to be expressed in a different way.

Apart from the standard proof rules of modal logic (*modus ponens* and *necessitation*), the system derived from this axiomatization contains also some unorthodox ones which are typical of hybrid logic.

Definition 6 (The logic) *Let* HSF *be the smallest set of formulas containing all the above axiom schemata and closed under application of the following rules:* [2]

$$(\text{MODUS PONENS}) \quad \frac{\alpha \to \beta, \alpha}{\beta}$$

$$(\Delta\text{–NECESSITATION}) \quad \frac{\alpha}{\Delta\alpha}$$

$$(\text{NAME}_{stat}) \quad \frac{j \to \beta}{\beta} \qquad (\text{NAME}_{sets}) \quad \frac{B \to \beta}{\beta}$$

$$(\mathsf{E}_\nabla\text{–ENRICHMENT}) \quad \frac{\mathsf{E}\,(i \wedge A \wedge \nabla(j \wedge B \wedge \alpha)) \to \beta}{\mathsf{E}\,(i \wedge A \wedge \nabla\alpha) \to \beta},$$

where $\alpha, \beta \in$ WFF, $i, j \in \mathrm{N}_{stat}$, $A, B \in \mathrm{N}_{sets}$, $\Delta \in \{K, \Box, \mathsf{A}\} \cup \{[f] \mid f \in \mathscr{F}\}$, $\nabla \in \{L, \Diamond, \mathsf{E}\} \cup \{\langle f \rangle \mid f \in \mathscr{F}\}$, *and* j, B *are* new *each time (i.e., do not occur in any other syntactic building block of the respective rule).*

The NAME and ENRICHMENT rules have to be used for proving an appropriate *Lindenbaum Lemma;* cf Blackburn, de Rijke, & Venema 2001, Lemma 7.25. This makes up the first step towards the following theorem.

Theorem 7 (Soundness and completeness) *Let* $\alpha \in$ WFF *be a formula. Then,* α *is valid in all SSFs iff it is* HSF*–derivable.*

While the proof of the soundness part of Theorem 7 is straightforward it is much harder to establish completeness. To this end, the canonical model of the system HSF has to be 'hybridised'; concerning this see Heinemann 2003 for the A–free fragment, and Heinemann 2005b for the full language (without functions). The desired model falsifying a given non-derivable formula, can finally be obtained as a certain space of partial functions, X, over that hybridised canonical model. In fact, the domain $\mathrm{dom}(h)$ of every function $h \in X$ is a maximal subset of the set $\mathscr{Q}$ of all equivalence classes of the accessibility relation induced by the modality K, with regard to the following two conditions:

1. $h([\Sigma]) \in [\Sigma]$ for all $[\Sigma] \in \mathrm{dom}(h)$, and
2. $h([\Sigma]) = h^{[\Theta]}_{[\Sigma]}(h([\Theta]))$ for all $[\Sigma], [\Theta] \in \mathrm{dom}(h)$ such that $[\Sigma] \preccurlyeq [\Theta]$;

here, the precedence relation $\preccurlyeq$ is defined by

$$[\Sigma] \preccurlyeq [\Theta] :\Longleftrightarrow \exists \Sigma' \in [\Sigma], \Theta' \in [\Theta] : \Sigma' \stackrel{\Box}{\longrightarrow} \Theta',$$

where Σ, Θ are points of the carrier set D of the hybridised canonical model and $\stackrel{\Box}{\longrightarrow}$ denotes the accessibility relation belonging to the effort modality $\Box$. We write $h_\Sigma := h([\Sigma])$ in case $h([\Sigma])$ exists. Furthermore, we let

- $U_{[\Sigma]} := \{h \in X \mid h_\Sigma \text{ exists}\}$, for all $\Sigma \in D$,
- $\mathscr{O} := \{U_{[\Sigma]} \mid \Sigma \in D\} \cup \{X, \emptyset\}$, and
- $V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \longrightarrow \mathscr{P}(X)$ be defined by

$$h \in V(c) :\Longleftrightarrow \begin{cases} c \in h_\Sigma \text{ for some } \Sigma \in D \\ \text{such that } h_\Sigma \text{ exists,} \end{cases}$$

for all $c \in \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets}$.

With that, we then have the following *Truth Lemma:*

Lemma 8 $\mathscr{S} := (X, \mathscr{O})$ *is a set frame (without functions) and* V *an* $\mathscr{S}$*–valuation. Moreover, letting* $\mathscr{M} := (X, \mathscr{O}, V)$ *we have that for all formulas* α*, functions* $h \in X$*, and points* $\Sigma \in D$ *such that* $h \in U_{[\Sigma]}$,

$$h, U_{[\Sigma]} \models_{\mathscr{M}} \alpha \iff \alpha \in h_\Sigma.$$

Lemma 8 yields completeness in case no functions are contained in the language; cf Heinemann 2005b, Sec. 3. But taking functions too into account, it is really possible to extend the just indicated framework correspondingly. To this end, we first note that a function $h \in X$ is already determined by its value for a single argument. And, vice versa, every $\Sigma \in D$ in fact induces a function $\in X$ passing through this point. Therefore, that function is denoted h^Σ; see Heinemann 2005b, Lemma 3.9. Now, let $f \in \mathscr{F}$ be given, and let h be an arbitrary element of X. Then there is some $\Theta \in D$ such that h_Θ is defined. Because of Axiom 21 the accessibility relation $\stackrel{[f]}{\longrightarrow}$ coming along with $[f]$ is serial. Hence there exists some $\Sigma \in D$ satisfying $h_\Theta \stackrel{[f]}{\longrightarrow} \Sigma$. We now define $F_f : X \longrightarrow X$ through

$$F_f(h) := h^\Sigma.$$

In this way, every $f \in \mathscr{F}$ is realized as a certain *functional*.

Lemma 9 F_f *is well-defined.*

Proof. We have to show that the definition of F_f is independent of Σ and Θ. Keeping the notations from above we first assume that both $h_\Theta \stackrel{[f]}{\longrightarrow} \Sigma$ and $h_\Theta \stackrel{[f]}{\longrightarrow} \Sigma'$ are valid. We know that there is some $i \in \mathrm{N}_{stat}$ contained in Σ. It follows

[2] The letter 'H' should remind one of *hybrid logic,* the letter 'S' of *set spaces,* and the letter 'F' of the presence of *functions*.

that $\langle f\rangle i \in h_\Theta$. Hence $[f]i \in h_\Theta$ holds because of Axiom 22. Consequently, $i \in \Sigma'$. The properties of the hybridised canonical model now imply $h^\Sigma = h^{\Sigma'}$; cf Heinemann 2005b, Sec. 3.

Secondly, let $h_{\Theta'}$ be defined as well. Moreover, take again some $i \in \mathrm{N}_{stat}$ contained in Σ. This time we utilize that there exists some $\Xi \in D$ such that $\Xi \xrightarrow{\Box} \Theta$ and $\Xi \xrightarrow{\Box} \Theta'$. Consequently, $\Diamond\langle f\rangle i \in \Xi$. With the aid of Axiom 23 we infer $\Box\langle f\rangle i \in \Xi$ from that. Hence $\langle f\rangle i \in h_{\Theta'}$. This means that some $\xrightarrow{[f]}$–successor $\tilde{\Sigma}$ of $h_{\Theta'}$ contains i. Therefore, $h^\Sigma = h^{\tilde{\Sigma}}$ follows. This shows that the the definition of F_f is also independent of Θ, as desired. □

Consequently, $\mathscr{M} := \big(X, \mathscr{O}, \{F_f \mid f \in \mathscr{F}\}, V\big)$ is an SSF. Thus, in order to complete the proof of Theorem 7 it remains to establish the *Truth Lemma* for the case involving functions. This makes up the crucial step of the completeness proof for the new logic.

Lemma 10 *Lemma 8 remains valid in case functions are integrated into the system.*

Proof. Let $\alpha = \langle f\rangle\beta$. We first prove the easier right-to-left direction. Assume that $\alpha \in h_\Sigma$ is valid. Then there exists some $\Theta \in D$ such that $h_\Sigma \xrightarrow{[f]} \Theta$ and $\beta \in \Theta$. Using the above notations we let $g := h^\Theta$ and get $\Theta = g_\Theta$ with that. It then follows that $g \in U_{[\Theta]}$. From the induction hypothesis we obtain $g, U_{[\Theta]} \models_{\mathscr{M}} \beta$. But $g = F_f(h)$, by definition. Thus, due to the last clause of Definition 2 we have that $h, U_{[\Sigma]} \models_{\mathscr{M}} \langle f\rangle\beta$.

Conversely, let $h, U_{[\Sigma]} \models_{\mathscr{M}} \langle f\rangle\beta$. By Definition 2, there are $g \in X$ and $\Theta \in D$ such that $g \in U_{[\Theta]}$, $F_f(h) = g$ and $g, U_{[\Theta]} \models_{\mathscr{M}} \beta$. By applying the induction hypothesis we obtain $\beta \in g_\Theta$ from that. Now, we must still show that $h_\Sigma \xrightarrow{[f]} g_\Theta$ is valid. Here is the place where Axiom 24 comes into play. But first of all note that we conclude from the condition $F_f(h) = g$ and the definition of F_f that there is some $\Xi \in D$ satisfying $h_\Sigma \xrightarrow{[f]} g_\Xi$. This enables us to bring up a nominal argument similar to the one in the proof of the previous lemma. To this end, take $i \in \mathrm{N}_{stat}$ and $A \in \mathrm{N}_{sets}$ contained in g_Θ. (Both nominals really exist.) It then follows that $\mathsf{E}(i \wedge A) \in h_\Sigma$. Since i is contained in g_Ξ as well, we get that $\langle f\rangle i$ too is contained h_Σ. Now, Axiom 24 can be applied, yielding $\langle f\rangle(i \wedge A) \in h_\Sigma$. Thus there exists some $\xrightarrow{[f]}$–successor Θ' of h_Σ containing $i \wedge A$. However, this formula acts like a 'proper' nominal, i.e., has a unique denotation (cf Heinemann 2005b, Lemma 3.5, and see the remark following Definition 2). As a consequence we get that $\Theta' = g_\Theta$. Hence $h_\Sigma \xrightarrow{[f]} g_\Theta$ actually holds. This proves the left-to-right direction. □

The following corollary is a consequence of Theorem 7 and Proposition 5; cf Blackburn, de Rijke, & Venema 2001, 7.29, for the case of standard hybrid logic..

Corollary 11 *Let* HSFC *be the system obtained from* HSF *by adding, for every* $f \in \mathscr{F}$, *the schema from Proposition 5. Then,* HSFC *is sound and complete with respect to the class of all SSFs with continuous functions.*

Corollary 11 tells us that certain elementary facts about continuous functions are HSFC–derivable, eg, the well-known property that the composition $f \circ g$ of two continuous functions f, g is continuous, too. (This property is expressed by the formula schema $\langle f\rangle\langle g\rangle A \rightarrow \Diamond K\langle f\rangle\langle g\rangle A$.)

Decidability

We now argue that both HSF and HSFC are decidable sets of formulas.

Theorem 12 (Decidability I) *The set of all* HSF*–derivable formulas is decidable.*

Proof. The result can be obtained by means of the method of *filtration* (with the intention to establish a certain finite model property of the logic). To this end, the approach taken for the logic without functions, cf Heinemann 2004 or, more detailedly, Heinemann 2005b, has to be refined accordingly.

In order to validate Axioms 20 – 24 in the filtrated model we must first of all extend the filter set. This extension concerns nominals and function symbols occurring in the formula α for which we want to find a finite model. In concrete terms, the following sets of formulas must be added to the set Σ_0 we started out with for the function-free language (cf Heinemann 2005b, Sec. 4):

$$\Sigma_1 := \{[f]\neg i, [f]i \mid f \in \mathscr{F} \text{ and } i \in \mathrm{N}_{stat} \text{ occur in } \alpha\}$$

$$\Sigma_2 := \left\{\Box[f]\neg i, \Box\langle f\rangle i \;\middle|\; \left(\begin{array}{c} f \in \mathscr{F} \text{ and } i \in \mathrm{N}_{stat} \\ \text{occur in } \alpha \end{array}\right)\right\},$$

and $\Sigma_3 := \Sigma_4 \cup \Sigma_5$, where

$$\Sigma_4 := \left\{\mathsf{A}\neg(i \wedge A) \;\middle|\; \left(\begin{array}{c} i \in \mathrm{N}_{stat} \text{ and } A \in \mathrm{N}_{sets} \\ \text{occur in } \alpha \end{array}\right)\right\}$$

and

$$\Sigma_5 := \left\{[f]\neg(i \wedge A) \;\middle|\; \left(\begin{array}{c} f \in \mathscr{F}, i \in \mathrm{N}_{stat} \text{ and} \\ A \in \mathrm{N}_{sets} \text{ occur in } \alpha \end{array}\right)\right\}.$$

We then perform the same operations on the resulting set of formulas as in the previous case. Thus the new filter set too is finite and subformula closed.

Secondly, we take the *minimal* filtration of all the accessibility relations of the canonical model (where α is realized at some point). We modify the structure obtained in this way with regard to the nominals and the operators $[f]$, respectively, which do not occur in α. Hence this modification does not affect the semantics of α. (The interpretation of a function symbol not occurring in α may be an arbitrary serial relation respecting the instances of Axioms 22 – 24 which are relevant to α, and the denotation of every such nominal is the empty set.)

The just indicated model surgery procedure provides for the desired validity of Axioms 22 – 24. (Since Axiom 20 is generally valid and Axiom 21 expresses seriality, which passes down to *every* filtration, we may restrict attention to these axioms, actually.) In fact, exploiting both the minimality of the filtration and the definition of the sets Σ_1, Σ_2 and Σ_3, respectively, we can directly calculate in case all the names and functions involved in the respective formula really occur in α; otherwise the proof is even simper.

The validity of the finite model property of the logic HSF with respect to a certain recursively enumerable set of models can now be inferred from the above in a standard way. (An initial segment of the natural numbers can be chosen as the carrier set of every model from that set, actually.) This gives us the desired decidability; cf Blackburn, de Rijke, & Venema 2001, 6.13. □

The method described in the proof of Theorem 12 can also be applied to our hybrid logic of continuous functions, yielding the decidability of HSFC in a similar manner.

Theorem 13 (Decidability II) *The set of all formulas that are* HSFC*–derivable, is decidable.*

Concerning the proof of this theorem it should be remarked that the filter set has to be extended once more, according to the axiom schema for continuity from Proposition 5.

Summary and future research

In this paper, we proposed a hybridised modal logic, designated HSF(C), for reasoning about knowledge and (continuous) functions. The main issues of the paper concern the finite axiomatizability, soundness and completeness, and decidability, of the set of all HSF(C)–theorems. We principally utilized the power of hybrid logic for proving these results, in particular, the immunity of certain naming techniques from type extensions of the named objects.

Due to the presence of functions in the underlying language it is very likely that the HSF–satisfiability problem is hard for EXPTIME; cf Blackburn & Spaan 1993, Theorem 4.5. However, determining the exact complexity is still an open problem which has to be solved by future research. (Note that we cannot reduce the satisfiability problem for dynamic logic (as it was accordingly done in Blackburn & Spaan 1993) to HSF(C)–satisfiability since we deal with total functions here.)

As it stands, our system is rather general and has to be refined for special purposes thus. This concerns, in particular, the representation of actions through functions (see the introductory section). Thus a more concrete synthesis of knowledge and action is demanded for our approach. (Note that combining formalisms for knowledge and action is still an actual field of research; see van Ditmarsch, van der Hoek, & Kooi 2003 and Baader *et al.* 2005, respectively, for recent examples from two more established approaches to this topic.) And the role of continuity in connection with actions then has to be studied more detailedly.

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