A note on comparing semantics for conditionals

Christoph Beierle
Dept. of Computer Science
FernUniversität in Hagen, Germany
christoph.beierle@fernuni-hagen.de

Gabriele Kern-Isberner
Dept. of Computer Science
Universität Dortmund, Germany
gabriele.kern-isberner@cs.uni-dortmund.de

Abstract

In this paper, we will study semantics that have been used for conditionals in the area of knowledge representation and reasoning: A purely qualitative semantics based on the popular *system-of-spheres semantics* of Lewis, an ordinal semantics making use of rankings, and a possibilistic semantics. As a common framework for the corresponding logics, we will use *institutions* which provide formal rigidity based on category theory, but leaves enough abstract freedom to formalize and compare quite different logics. We will show that the conditional semantics mentioned above are logically similar, yet each semantics allows semantical subtleties.

Introduction

As a part of the subjective belief state of an agent, conditional statements *if* A *then* B, formally denoted by $(B|A)$, usually have interpretations that are quite different from that of material implications $A \Rightarrow B$. Conditionals are more commonly used to express intensional and meaningful relationships between antecedent, A, and consequent, B, making extensional truth valuations via $A \Rightarrow B \equiv \neg A \vee B$ useless. This is quite obvious in the case of *counterfactual conditionals* (Lewis 1973) the antecedent of which is known to be false, so their logical truth value (when interpreted as a material implication) would be *true*. However, whereas the counterfactual *If Christmas were in July, we wouldn't have snow on Christmas* would be considered an acceptable statement in the temperate zones of the northern hemisphere, the counterfactuals *If Christmas were in July, the law of gravity would no longer hold*, or even worse, *If Christmas were in July, then Christmas were in November* would be hardly accepted by any reasonable person.

So, conditionals need a richer semantical environment than classical bivalued logics to be interpreted adequately. Besides counterfactuals, there are other types of conditionals with peculiarities; a good logical overview can be found in (Nute 1980). In the area of knowledge representation, conditionals have a broad range of application: They can be considered as formal representations of default rules (Goldszmidt & Pearl 1996), or, on a meta level, as encodings of inference rules for nonmonotonic reasoning or of revision strategies for belief revision (Kern-Isberner 2001).

In this paper, we will study semantics that have been broadly used for conditionals in all these areas: A purely qualitative semantics based on the popular *system-of-spheres semantics* of Lewis (Lewis 1973), an ordinal semantics making use of rankings (Spohn 1988), and a possibilistic semantics (Benferhat, Dubois, & Prade 1999). In particular, we will be interested in elaborating abstract logical relationships between these semantics. To this aim, we will use the *institutions* which Goguen and Burstall introduced as a general formal framework for logical systems (Goguen & Burstall 1992). An institution formalizes the informal notion of a logical system, including syntax, semantics, and the relation of satisfaction between them. The latter poses the major requirement for an institution: that the satisfaction relation is consistent under the change of notation. This paper continues work on comparing logics for knowledge representation and reasoning with very different syntax and semantics, including propositional and probabilistic logics (cf. (Beierle & Kern-Isberner 2002; 2003)).

The organization of the paper is as follows: The next two sections are dedicated to explaining the two basic concepts that this paper deals with, namely institutions and semantics for conditionals. As the main contributions of this paper, we continue by formalizing purely qualitative, ordinal and possibilistic logics for conditionals as institutions, and study connections between them by morphisms. Finally, we relate these results to work done previously, and conclude the paper by a summary.

Institutions

In this section, we will describe briefly the framework of institutions we will be working with. As institutions are formalized by using category theory, we will also very briefly recall some basic notions of category theory; for more information about categories, see e.g. (Herrlich & Strecker 1973). To give an example, we present the institution of propositional logic the components of which will be used throughout the paper. Moreover, we explain how institutions can be related to each other by institution morphisms.

Basic Definitions and Notations

If C is a category, $|C|$ denotes the objects of C and $/C/$ its morphisms; for both objects $c \in |C|$ and morphisms

$\varphi \in /C/$, we also write just $c \in C$ and $\varphi \in C$, respectively. C^{op} is the opposite category of C, with the direction of all morphisms reversed. The composition of two functors $F : C \to C'$ and $G : C' \to C''$ is denoted by $G \circ F$ (first apply F, then G). For functors $F, G : C \to C'$, a natural transformation η from F to G, denoted by $\eta : F \Longrightarrow G$, assigns to each object $c \in |C|$ a morphism $\eta_c : F(C) \to G(C) \in /C'/$ such that for every morphism $\varphi : c \to d \in /C/$ we have $\eta_d \circ F(\varphi) = G(\varphi) \circ \eta_c$. $\mathcal{SET}$ and $\mathcal{CAT}$ denote the categories of sets and of categories, respectively.

The central definition of an institution (Goguen & Burstall 1992) is the following:

Definition 1 An institution is a quadruple $Inst = \langle\, Sig,\, Mod,\, Sen,\, \models \,\rangle$ with a category Sig of signatures as objects, a functor $Mod : Sig \to \mathcal{CAT}^{op}$ yielding the category of Σ-models for each signature Σ, a functor $Sen : Sig \to \mathcal{SET}$ yielding the sentences over a signature, and a $|Sig|$-indexed relation $\models_\Sigma \;\subseteq\; |Mod(\Sigma)| \times Sen(\Sigma)$ such that for each signature morphism $\varphi : \Sigma \to \Sigma' \in /Sig/$, for each $m' \in |Mod(\Sigma')|$, and for each $f \in Sen(\Sigma)$ the following *satisfaction condition* holds:

$$m' \models_{\Sigma'} Sen(\varphi)(f) \quad \text{iff} \quad Mod(\varphi)(m') \models_\Sigma f$$

For sets F, G of Σ-sentences and a Σ-model m we write $m \models_\Sigma F$ iff $m \models_\Sigma f$ for all $f \in F$. The satisfaction relation is lifted to semantical entailment $\models_\Sigma$ between sentences by defining $F \models_\Sigma G$ iff for all Σ-models m with $m \models_\Sigma F$ we have $m \models_\Sigma G$. $F^\bullet = \{f \in Sen(\Sigma) \mid F \models_\Sigma f\}$ is called the *closure* of F, and F is *closed* if $F = F^\bullet$. The closure operator fulfils the *closure lemma* $\varphi(F^\bullet) \subseteq \varphi(F)^\bullet$ and various other nice properties like $\varphi(F^\bullet)^\bullet = \varphi(F)^\bullet$ or $(F^\bullet \cup G)^\bullet = (F \cup G)^\bullet$. A consequence of the closure lemma is that entailment is preserved under change of notation carried out by a signature morphism, i.e. $F \models_\Sigma G$ implies $\varphi(F) \models_{\varphi(\Sigma)} \varphi(G)$ (but not vice versa).

The Institution of Propositional Logic

The components of the institution $Inst_\mathcal{B} = \langle\, Sig_\mathcal{B},\, Mod_\mathcal{B},\, Sen_\mathcal{B},\, \models_\mathcal{B} \,\rangle$ of classical propositional logic will be defined in the following.

Signatures: $Sig_\mathcal{B}$ is the category of propositional signatures. A propositional signature $\Sigma \in |Sig_\mathcal{B}|$ is a (finite) set of propositional variables, $\Sigma = \{a_1, a_2, \ldots\}$. A propositional signature morphism $\varphi : \Sigma \to \Sigma' \in /Sig_\mathcal{B}/$ is a function mapping propositional variables to propositional variables.

Models: For each signature $\Sigma \in Sig_\mathcal{B}$, $Mod_\mathcal{B}(\Sigma)$ contains the set of all propositional interpretations for Σ, i.e.

$$|Mod_\mathcal{B}(\Sigma)| \;=\; \{I \mid I : \Sigma \to Bool\}$$

where $Bool = \{true, false\}$. Due to its simple structure, the only morphisms in $Mod_\mathcal{B}(\Sigma)$ are the identity morphisms. For each signature morphism $\varphi : \Sigma \to \Sigma' \in Sig_\mathcal{B}$, we define the functor $Mod_\mathcal{B}(\varphi) : Mod_\mathcal{B}(\Sigma') \to Mod_\mathcal{B}(\Sigma)$ by $(Mod_\mathcal{B}(\varphi)(I'))(a_i) \;:=\; I'(\varphi(a_i))$ where $I' \in Mod_\mathcal{B}(\Sigma')$ and $a_i \in \Sigma$.

Sentences: For each signature $\Sigma \in Sig_\mathcal{B}$, the set $Sen_\mathcal{B}(\Sigma)$ contains the usual propositional formulas constructed from the propositional variables in Σ and the logical connectives $\wedge$ (and), $\vee$ (or), and $\neg$ (not).

For each signature morphism $\varphi : \Sigma \to \Sigma' \in Sig_\mathcal{B}$, the function $Sen_\mathcal{B}(\varphi) : Sen_\mathcal{B}(\Sigma) \to Sen_\mathcal{B}(\Sigma')$ is defined by straightforward inductive extension on the structure of the formulas; e.g., $Sen_\mathcal{B}(\varphi)(a_i) = \varphi(a_i)$ and $Sen_\mathcal{B}(\varphi)(A \wedge B) = Sen_\mathcal{B}(\varphi)(A) \wedge Sen_\mathcal{B}(\varphi)(B)$. In the following, we will abbreviate $Sen_\mathcal{B}(\varphi)(A)$ by just writing $\varphi(A)$.

In order to simplify notations, we will often replace conjunction by juxtaposition and indicate negation of a formula by barring it, i.e. $AB = A \wedge B$ and $\overline{A} = \neg A$. An *atomic formula* is a formula consisting of just a propositional variable, a *literal* is a positive or a negated atomic formula, an *elementary conjunction* is a conjunction of literals, and a *complete conjunction* is an elementary conjunction containing each atomic formula either in positive or in negated form. Ω_Σ denotes the set of all complete conjunctions over a signature Σ; if Σ is clear from the context, we may drop the index Σ. Note that there is an obvious bijection between $|Mod_\mathcal{B}(\Sigma)|$ and Ω_Σ, associating with $I \in |Mod_\mathcal{B}(\Sigma)|$ the complete conjunction $\omega_I \in \Omega_\Sigma$ in which an atomic formula $a_i \in \Sigma$ occurs in positive form iff $I(a_i) = true$.

Satisfaction relation: For any $\Sigma \in |Sig_\mathcal{B}|$, the satisfaction relation

$$\models_{\mathcal{B},\Sigma} \;\subseteq\; |Mod_\mathcal{B}(\Sigma)| \times Sen_\mathcal{B}(\Sigma)$$

is defined as expected for propositional logic, e.g. $I \models_{\mathcal{B},\Sigma} a_i$ iff $I(a_i) = true$ and $I \models_{\mathcal{B},\Sigma} A \wedge B$ iff $I \models_{\mathcal{B},\Sigma} A$ and $I \models_{\mathcal{B},\Sigma} B$ for $a_i \in \Sigma$ and $A, B \in Sen_\mathcal{B}(\Sigma)$.

Proposition 2 *$Inst_\mathcal{B} = \langle\, Sig_\mathcal{B},\, Mod_\mathcal{B},\, Sen_\mathcal{B},\, \models_\mathcal{B} \,\rangle$ is an institution.*

A proof of this proposition is straightforward since the satisfaction condition $I' \models_{\mathcal{B},\Sigma'} \varphi(A)$ iff $Mod_\mathcal{B}(\varphi)(I') \models_{\mathcal{B},\Sigma} A$ holds by easy induction on the structure of A.

Example 3 Let $\Sigma = \{s, t, u\}$ and $\Sigma' = \{a, b, c\}$ be two propositional signatures with the atomic propositions *s – being a scholar*, *t – being not married*, *u – being single* and *a – being a student*, *b – being young*, *c – being unmarried*. Let I' be the Σ'-model with $I'(a) = true$, $I'(b) = true$, $I'(c) = false$. Let $\varphi : \Sigma \to \Sigma' \in Sig_\mathcal{B}$ be the signature morphism with $\varphi(s) = a$, $\varphi(t) = c$, $\varphi(u) = c$. The functor $Mod_\mathcal{B}(\varphi)$ takes I' to the Σ-model $I := Mod_\mathcal{B}(\varphi)(I')$, yielding $I(s) = I'(a) = true$, $I(t) = I'(c) = false$, $I(u) = I'(c) = false$. ■

Note that in the example, φ is neither surjective nor injective. φ not being surjective makes the functor $Mod_\mathcal{B}(\varphi)$ a forgetful functor – any information about *b* (being young) in I' is forgotten in I. φ not being injective implies that any two propositional variables from Σ mapped to the same element in Σ' are both being identified with the same proposition; thus, under the forgetful functor $Mod_\mathcal{B}(\varphi)$, the interpretation of *t* (*being not married*) and *u* (*being single*) will always be equivalent since $\varphi(t) = \varphi(u)$.

Institution Morphisms

An institution morphism Φ expresses a relation between two institutions *Inst* und *Inst'* such that the satisfaction condition of *Inst* may be computed by the satisfaction condition of *Inst'* if we translate it according to Φ. The translation is done by relating every *Inst*-signature Σ to an *Inst'*-signature Σ', each Σ'-sentence to a Σ-sentence, and each Σ-model to a Σ'-model.

Definition 4 Let $Inst, Inst'$ be two institutions, $Inst = \langle\, Sig, Mod, Sen, \models \,\rangle$ and $Inst' = \langle\, Sig', Mod', Sen', \models' \,\rangle$. An *institution morphism* Φ from *Inst* to *Inst'* is a triple $\langle\, \phi, \alpha, \beta \,\rangle$ with a functor $\phi : Sig \rightarrow Sig'$, a natural transformation $\alpha : Sen' \circ \phi \Longrightarrow Sen$, and a natural transformation $\beta : Mod \Longrightarrow Mod' \circ \phi$ such that for each $\Sigma \in |\,Sig\,|$, for each $m \in |\,Mod(\Sigma)\,|$, and for each $f' \in Sen'(\phi(\Sigma))$ the following *satisfaction condition (for institution morphisms)* holds:

$$m \models_\Sigma \alpha_\Sigma(f') \quad \text{iff} \quad \beta_\Sigma(m) \models'_{\phi(\Sigma)} f' \qquad (1)$$

Semantics for conditionals

Various types of models have been proposed to interpret conditionals $(B|A)$ (with propositional formulas A, B) adequately within a logical system (cf. e.g. (Nute 1980)). Many of them are based on considering possible worlds which can be thought of as being represented by classical logical interpretations $|\,Mod_\mathcal{B}(\Sigma)\,|$, or complete conjunctions $\omega \in \Omega$, respectively. One of the most prominent approaches is the *system-of-spheres* model of Lewis (Lewis 1973) which makes use of a notion of similarity between possible worlds. This idea of comparing worlds and evaluating conditionals with respect to the "nearest" or "best" worlds (which are somehow selected) is common to very many approaches in conditional logics.

From a purely qualitative point of view, proper models of conditionals are provided by total preorders over classical propositional interpretations, or possible worlds, respectively. Possible worlds are ordered according to their *plausibility*. By convention, the least worlds are the most plausible worlds. We will also use the infix notation $\omega_1 \preceq_R \omega_2$ instead of $(\omega_1, \omega_2) \in R$. As usual, we introduce the $\prec_R$-relation by saying that $\omega_1 \prec_R \omega_2$ iff $\omega_1 \preceq_R \omega_2$ and not $\omega_2 \preceq_R \omega_1$. Furthermore, $\omega_1 \approx_R \omega_2$ means that both $\omega_1 \preceq_R \omega_2$ and $\omega_2 \preceq_R \omega_1$ hold.

Each total preorder R induces a partitioning $\Omega_0, \Omega_1, \ldots$ of Ω, such that all worlds in the same partitioning subset are considered equally plausible ($\omega_1 \approx_R \omega_2$ for $\omega_1, \omega_2 \in \Omega_j$), and whenever $\omega_1 \in \Omega_i$ and $\omega_2 \in \Omega_k$ with $i < k$, then $\omega_1 \prec_R \omega_2$. Let $Min(R)$ denote the set of R-minimal worlds in Ω, i.e.

$$Min(R) = \Omega_0 = \{\omega_0 \in \Omega \mid \omega_0 \preceq_R \omega \text{ for all } \omega \in \Omega\}$$

Moreover, a total preorder on propositional formulas $A, B \in Sen_\mathcal{B}(\Sigma)$ is defined by

$$\begin{array}{lll} A \preceq_R B & \text{iff} & \text{for all } \omega_2 \in \Omega \text{ with } \omega_2 \models_{\mathcal{B},\Sigma} B \\ & & \text{there exists } \omega_1 \in \Omega \text{ with } \omega_1 \models_{\mathcal{B},\Sigma} A \\ & & \text{such that } \omega_1 \preceq_R \omega_2 \end{array}$$

So, A is considered to be at least as plausible as B (with respect to R) iff the most plausible worlds satisfying A are at least as plausible as any world satisfying B. Finally, a total preorder R *represents* or *accepts* a conditional $(B|A)$ iff $AB \prec_R A\overline{B}$, i.e. iff the verification of the conditional (AB) is found more plausible than its falsification $(A\overline{B})$.

There are other, more fine-grained semantics for conditionals which aim at making the vague notion of plausibility preorders more precise by using numbers to compare different degrees of "plausibility" between the verification and the falsification of a conditional. Here, two of the most popular approaches make use of ordinal rankings and of possibility theory, respectively. We will sketch these semantics in the following.

The basic idea of *ordinal conditional functions (OCFs)* is to specify the partitioning sets defined by total preorders by level numbers. OCFs are simply functions $\kappa : \Omega \rightarrow \mathbb{N}$ with $\kappa^{-1}(0) \neq \emptyset$. The smaller $\kappa(\omega)$, the more plausible is the world ω. So ordinal rankings actually express a *degree of disbelief* or *surprise* to observe the corresponding world. For propositional formulas $A, B \in Sen_\mathcal{B}(\Sigma)$, we have $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$, so that $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$. Here, conditionals are given semantics by defining that an OCF κ *accepts* $(B|A)$ iff $\kappa(AB) < \kappa(A\overline{B})$, i.e. if it is a greater surprise to observe $A\overline{B}$ than AB.

A *possibility distribution* is a function $\pi : \Omega \rightarrow [0, 1]$ (Dubois & Prade 1994)). Each possibility distribution induces a *possibility measure* on propositional formulas which will also be denoted by π: For each $A \in Sen_\mathcal{B}(\Sigma)$, $\pi(A) = \max_{\omega \models A} \pi(\omega)$. Then it holds that $\pi(A \vee B) = \max\{\pi(A), \pi(B)\}$, and $\pi(A \wedge B) \leqslant \min\{\pi(A), \pi(B)\}$ for any two propositional formulas $A, B \in Sen_\mathcal{B}(\Sigma)$. Here, a conditional $(B|A)$ is *accepted by* π iff $\pi(AB) > \pi(A\overline{B})$, i.e. if its verification is more *possible* than its falsification.

Institutions of conditional logics

In this section, we will formalize institutions for the conditional logics sketched above. More precisely, we will define the institution $Inst_\mathcal{K}$ of purely qualitative conditionals, the institution $Inst_\mathcal{O}$ of ordinal conditionals, and the institution $Inst_\Pi$ of possibilistic conditionals. The signatures of all these institutions will be the same, namely the signature $Sig_\mathcal{B}$ of propositional logic. So, all these logics will use a common vocabulary, and, as we will see, also a common syntax. This allows us to focus on semantical peculiarities.

The institution $Inst_\mathcal{K}$ of purely qualitative conditionals

In the following, we will describe the components of the institution $Inst_\mathcal{K} = \langle\, Sig_\mathcal{K}, Mod_\mathcal{K}, Sen_\mathcal{K}, \models_\mathcal{K} \,\rangle$ of purely qualitative conditionals. This section recalls results from (Beierle & Kern-Isberner 2002).

Signatures: $Sig_\mathcal{K}$ is identical to the category of propositional signatures, i.e. $Sig_\mathcal{K} = Sig_\mathcal{B}$.

Models: In correspondence with Lewis' system-of-spheres semantics, we will consider the models $R \in Mod_\mathcal{K}(\Sigma)$ to be total preorders on Ω, ordering the possible worlds according to their plausibility, or similarity with the actual world:

$$|\,Mod_{\mathcal{K}}(\Sigma)\,| \;=\; \{R \mid R \text{ is a total preorder on } |\,Mod_{\mathcal{B}}(\Sigma)\,|\}$$

We only consider the identity morphisms in $Mod_{\mathcal{K}}(\Sigma)$ for this paper.

For each signature morphism $\varphi : \Sigma \to \Sigma'$, we define a functor $Mod_{\mathcal{K}}(\varphi) : Mod_{\mathcal{K}}(\Sigma') \to Mod_{\mathcal{K}}(\Sigma)$ by mapping a (total) preorder R' over $Mod_{\mathcal{B}}(\Sigma')$ to a (total) preorder $Mod_{\mathcal{K}}(\varphi)(R')$ over $Mod_{\mathcal{B}}(\Sigma)$ in the following way:

$$\omega_1 \preceq_{Mod_{\mathcal{K}}(\varphi)(R')} \omega_2 \quad \text{iff} \quad \varphi(\omega_1) \preceq_{R'} \varphi(\omega_2) \qquad (2)$$

Note that on the left hand side of (2) the complete conjunctions ω_1 and ω_2 are viewed as models in $Mod_{\mathcal{B}}(\Sigma)$, whereas on the right hand side they are sentences in $Sen_{\mathcal{B}}(\Sigma)$.

It is straightforward to check that $Mod_{\mathcal{K}}(\varphi)(R')$ is a total preorder (the corresponding properties are all directly inherited by R'), so indeed $Mod_{\mathcal{K}}(\varphi)(R') \in Mod_{\mathcal{K}}(\Sigma)$. The connection between R' and $Mod_{\mathcal{K}}(\varphi)(R')$ defined by (2) can also be shown to hold for propositional sentences instead of worlds:

Lemma 5 *Let $A, B \in Sen_{\mathcal{B}}(\Sigma)$. Then $A \preceq_{Mod_{\mathcal{K}}(\varphi)(R')} B$ iff $\varphi(A) \preceq_{R'} \varphi(B)$.*

Sentences: For each signature Σ, the set $Sen_{\mathcal{K}}(\Sigma)$ contains (propositional) *conditionals* of the form $(B|A)$ where $A, B \in Sen_{\mathcal{B}}(\Sigma)$ are propositional formulas from $Inst_{\mathcal{B}}$. For $\varphi : \Sigma \to \Sigma'$, the extension $Sen_{\mathcal{K}}(\varphi)$ is defined as usual by $Sen_{\mathcal{K}}(\varphi)((B|A)) = (\varphi(B)|\varphi(A))$.

Satisfaction relation: The satisfaction relation $\models_{\mathcal{K},\Sigma} \subseteq |\,Mod_{\mathcal{K}}(\Sigma)\,| \times Sen_{\mathcal{K}}(\Sigma)$ is defined, for any $\Sigma \in |\,Sig_{\mathcal{K}}\,|$, by

$$R \models_{\mathcal{K},\Sigma} (B|A) \quad \text{iff} \quad AB \prec_R A\overline{B}$$

Therefore, a conditional $(B|A)$ is satisfied (or accepted) by the plausibility preorder R iff its confirmation AB is more plausible than its refutation $A\overline{B}$.

Using Lemma 5, it is easy to prove the following proposition.

Proposition 6 *$Inst_{\mathcal{K}} = \langle\, Sig_{\mathcal{K}},\, Mod_{\mathcal{K}},\, Sen_{\mathcal{K}},\, \models_{\mathcal{K}} \,\rangle$ is an institution.*

Example 7 We continue our student-example in this qualitative conditional environment, so let Σ, Σ', φ be as defined in Example 3. Let R' be the following total preorder on Ω':

$$\begin{aligned} R': \qquad & \overline{a}\overline{b}\overline{c} \prec_{R'} abc \approx_{R'} \overline{a}bc \prec_{R'} ab\overline{c}, \\ & a\overline{b}\overline{c} \approx_{R'} a\overline{b}c \approx_{R'} a\overline{b}\overline{c} \approx_{R'} \overline{a}b\overline{c} \approx_{R'} \overline{a}\overline{b}c \end{aligned}$$

Now, for instance $R' \models_{\mathcal{K},\Sigma'} (\overline{a}|\top)$ since $\top\overline{a} \equiv \overline{a}$, $\top\overline{\overline{a}} \equiv a$, and $\overline{a} \prec_{R'} a$. Thus, under R', it is more plausible to be not a student than to be a student. Furthermore, $R' \models_{\mathcal{K},\Sigma'} (c|a)$ – *students* are supposed to be *unmarried* since under R', ac is more plausible than $a\overline{c}$.

Under $Mod_{\mathcal{K}}(\varphi)$, R' is mapped onto $R = Mod_{\mathcal{K}}(\varphi)(R')$ where R is the following total preorder on Ω:

$$\begin{aligned} R: \qquad & \overline{s}\overline{t}\overline{u} \prec_R \overline{s}tu \approx_R stu \prec_R s\overline{t}\overline{u} \prec_R st\overline{u}, \\ & st\overline{u} \approx_R s\overline{t}u \approx_R \overline{s}t\overline{u} \approx_R \overline{s}\overline{t}u \end{aligned}$$

As expected, the conditionals $(t|s)$ and $(u|s)$, both corresponding to $(c|a)$ in $Sen_{\mathcal{K}}(\Sigma')$ under φ, are satisfied by R – here, *scholars* are supposed to be both *not married* and *single*. ■

The institution $Inst_{\mathcal{O}}$ of ordinal conditionals

Signatures and Sentences: As in the previous section, we choose propositional signatures: $Sig_{\mathcal{O}} = Sig_{\mathcal{B}}$. For the sentences functor, we make use of the constructions for purely qualitative conditionals which provide a formal language for conditionals; hence $Sen_{\mathcal{O}} = Sen_{\mathcal{K}}$.

Models: The models of this institution are Spohn's *ordinal conditional functions* (*OCFs*) (Spohn 1988) $\kappa : \Omega_\Sigma \to \mathbb{N}$. Hence

$$|\,Mod_{\mathcal{O}}(\Sigma)\,| \;=\; \{\kappa : \Omega_\Sigma \to \mathbb{N} \mid \kappa \text{ is an OCF}\}$$

For each signature morphism $\varphi : \Sigma \to \Sigma'$, we define a functor $Mod_{\mathcal{O}}(\varphi) : Mod_{\mathcal{O}}(\Sigma') \to Mod_{\mathcal{O}}(\Sigma)$ by mapping an OCF κ' over $Mod_{\mathcal{O}}(\Sigma')$ to an OCF $Mod_{\mathcal{O}}(\varphi)(\kappa')$ over $Mod_{\mathcal{O}}(\Sigma)$ in the following way:

$$Mod_{\mathcal{O}}(\varphi)(\kappa')(\omega) = \kappa'(\varphi(\omega)) = \min_{\omega' \models \varphi(\omega)} \kappa'(\omega') \qquad (3)$$

It is straightforward to generalize this to proropositional formulas $A \in Sen_{\mathcal{B}}(\Sigma)$:

$$Mod_{\mathcal{O}}(\varphi)(\kappa')(A) = \kappa'(\varphi(A)) \qquad (4)$$

Satisfaction relation: The satisfaction relation $\models_{\mathcal{O},\Sigma} \subseteq |\,Mod_{\mathcal{O}}(\Sigma)\,| \times Sen_{\mathcal{O}}(\Sigma)$ is defined, for any $\Sigma \in |\,Sig_{\mathcal{O}}\,|$, by

$$\kappa \models_{\mathcal{O},\Sigma} (B|A) \quad \text{iff} \quad \kappa(AB) < \kappa(A\overline{B})$$

Therefore, a conditional $(B|A)$ is satisfied (or accepted) by the ordinal conditional function κ iff its confirmation AB is less surprizing than its refutation $A\overline{B}$. Using (4), we are now able to prove that these components make up an institution.

Proposition 8 *$Inst_{\mathcal{O}} = \langle\, Sig_{\mathcal{O}},\, Mod_{\mathcal{O}},\, Sen_{\mathcal{O}},\, \models_{\mathcal{O}} \,\rangle$ is an institution.*

The institution $Inst_{\Pi}$ of possibilistic conditionals

We start with specifying the language of this institution.

Signatures and Sentences: As for ordinal conditionals, we have $Sig_{\Pi} = Sig_{\mathcal{B}}$ and $Sen_{\Pi} = Sen_{\mathcal{K}}$.

Models: The models of this institution are possibility distributions $\pi : \Omega_\Sigma \to [0,1]$, i.e.

$$|\,Mod_{\Pi}(\Sigma)\,| \;=\; \{\pi : \Omega_\Sigma \to [0,1]\}$$

For each signature morphism $\varphi : \Sigma \to \Sigma'$, we define a functor $Mod_{\Pi}(\varphi) : Mod_{\Pi}(\Sigma') \to Mod_{\Pi}(\Sigma)$ by mapping a possibility distribution π' over $Mod_{\Pi}(\Sigma')$ to a possibility distribution $Mod_{\Pi}(\varphi)(\pi')$ over $Mod_{\Pi}(\Sigma)$ in the following way:

$$Mod_{\Pi}(\varphi)(\pi')(\omega) = \pi'(\varphi(\omega)) \qquad (5)$$

It is obvious that $Mod_{\Pi}(\varphi)(\pi')$ is a possibility distribution.

For a propositional formula $A \in Sen_{\mathcal{B}}(\Sigma)$, we have $Mod_{\Pi}(\varphi)(\pi')(A) = \pi'(\varphi(A))$.

Satisfaction relation: The satisfaction relation $\models_{\Pi,\Sigma} \subseteq |\mathit{Mod}_\Pi(\Sigma)| \times \mathit{Sen}_\Pi(\Sigma)$ is defined, for any $\Sigma \in |\mathit{Sig}_\Pi|$, by

$$\pi \models_{\Pi,\Sigma} (B|A) \quad \text{iff} \quad \pi(AB) > \pi(A\overline{B})$$

Therefore, a conditional $(B|A)$ is satisfied (or accepted) by the possibility distribution π iff its confirmation AB is more possible than its refutation $A\overline{B}$.

Proposition 9 *$\mathit{Inst}_\Pi = \langle \mathit{Sig}_\Pi, \mathit{Mod}_\Pi, \mathit{Sen}_\Pi, \models_\Pi \rangle$ is an institution.*

Relationships between different conditional institutions

Since $\mathit{Inst}_\mathcal{K}$, $\mathit{Inst}_\mathcal{O}$, and Inst_Π, all have the same category $\mathit{Sig}_\mathcal{B}$ of signatures, a natural choice for the signature translation component ϕ in any morphism between these institutions is the identity $id_{\mathit{Sig}_\mathcal{B}}$ which we will use in the following. Moreover, since all sentences functors are also identical to $\mathit{Sen}_\mathcal{K}$, any morphism between these institutions should use the identical natural transformation

$$\alpha_{id} : \mathit{Sen}_\mathcal{K} \Longrightarrow \mathit{Sen}_\mathcal{K} \qquad \alpha_{id,\Sigma}((B|A)) = (B|A) \ .$$

We will relate $\mathit{Inst}_\mathcal{K}$ to each of $\mathit{Inst}_\mathcal{O}$ and Inst_Π; relations between $\mathit{Inst}_\mathcal{O}$ and Inst_Π can be obtained directly by combining the results shown.

Relating $\mathit{Inst}_\mathcal{K}$ and Inst_Π

In this subsection, we will investigate whether the introduction of numbers in possibility theory makes a substantial difference to interpreting conditionals in a purely qualitative way, as is done in $\mathit{Inst}_\mathcal{K}$.

We are left with the problem of specifying natural transformations $\beta_1 : \mathit{Mod}_\mathcal{K} \Longrightarrow \mathit{Mod}_\Pi$ and $\beta_2 : \mathit{Mod}_\Pi \Longrightarrow \mathit{Mod}_\mathcal{K}$ between the models.

The models of $\mathit{Inst}_\mathcal{K}$ are total preorders on the set of possible worlds Ω_Σ and hence correspond to *comparative possibility distributions* (see (Benferhat, Dubois, & Prade 1999)). Obviously, each possibility distribution π on Ω_Σ can be mapped onto a total preorder

$$R_\pi \in \mathit{Mod}_\mathcal{K}(\Sigma), \quad \omega_1 \preceq_R \omega_2 \text{ iff } \pi(\omega_1) \geqslant \pi(\omega_2).$$

Conversely, if R is a total preorder in $\mathit{Mod}_\mathcal{K}(\Sigma)$ defining a partitioning $\Omega_0, \Omega_1, \ldots, \Omega_n$ on Ω_Σ with *Min*$(R) = \Omega_0$, we choose a sequence $a_0 = 1 > a_1 > \ldots > a_n > 0$ of real numbers and define a (normalized) possibility distribution $\pi_R \in \mathit{Mod}_\Pi(\Sigma)$, $\pi_R(\omega) = a_i$ iff $\omega \in \Omega_i$. Hence plausibility is identified with possibility. Each possibility distribution determines a total preorder in a unique way, but note that a whole bunch of possibility distributions give rise to the same total preorder.

$$\begin{array}{llll} \beta_{\mathcal{K}/\Pi} : \mathit{Mod}_\mathcal{K} \Longrightarrow \mathit{Mod}_\Pi & \beta_{\mathcal{K}/\Pi,\Sigma}(R) & = & \pi_R \\ \beta_{\Pi/\mathcal{K}} : \mathit{Mod}_\Pi \Longrightarrow \mathit{Mod}_\mathcal{K} & \beta_{\Pi/\mathcal{K},\Sigma}(\pi) & = & R_\pi \end{array} \qquad (6)$$

are both natural transformations establishing morphisms between both institutions:

Proposition 10 *Let the natural transformations $\beta_{\mathcal{K}/\Pi}$ and $\beta_{\Pi/\mathcal{K}}$ be as defined in (6). Then both $\langle id_{\mathit{Sig}_\mathcal{B}}, \alpha_{id}, \beta_{\mathcal{K}/\Pi} \rangle : \mathit{Inst}_\mathcal{K} \longrightarrow \mathit{Inst}_\Pi$ and $\langle id_{\mathit{Sig}_\mathcal{B}}, \alpha_{id}, \beta_{\Pi/\mathcal{K}} \rangle : \mathit{Inst}_\Pi \longrightarrow \mathit{Inst}_\mathcal{K}$ are institution morphisms. Moreover, $\langle id_{\mathit{Sig}_\mathcal{B}}, \alpha_{id}, \beta_{\Pi/\mathcal{K}} \rangle$ is the only such morphism.*

Relating $\mathit{Inst}_\mathcal{O}$ and $\mathit{Inst}_\mathcal{K}$

We focus our investigations on relating models of these institutions by natural transformations.

Going from $\mathit{Inst}_\mathcal{O}$ to $\mathit{Inst}_\mathcal{K}$, we have to transform ordinal conditional functions into total preorders so that the satisfaction condition for morphisms

$$\kappa \models_{\mathcal{O},\Sigma} (B|A) \quad \text{iff} \quad \beta_{\mathcal{O}/\mathcal{K},\Sigma}(\kappa) \models_{\mathcal{K},\Sigma} (B|A)$$

holds with a natural transformation $\beta_{\mathcal{O}/\mathcal{K}} : \mathit{Mod}_\mathcal{O} \Longrightarrow \mathit{Mod}_\mathcal{K}, \beta_{\mathcal{O}/\mathcal{K},\Sigma}(\kappa) = R_\kappa$. It is obvious to define R_κ by

$$\omega_1 \preceq_{R_\kappa} \omega_2 \quad \text{iff} \quad \kappa(\omega_1) \leqslant \kappa(\omega_2). \qquad (7)$$

This is lifted easily to the level of propositions, i.e. we have $A \preceq_{R_\kappa} B$ iff $\kappa(A) \leqslant \kappa(B)$, since both κ and R_κ evaluate formulas on minimal worlds. This implies $A \prec_{R_\kappa} B$ iff $\kappa(A) \leqslant \kappa(B)$, which is important for checking conditionals.

In a similar way as for possibilistic conditionals (see Proposition 10), it can be shown that the satisfaction condition above is equivalent to stating $\omega_1 \preceq_{R_\kappa} \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$. Therefore, we have proved the following

Proposition 11 *Let $\beta_{\mathcal{O}/\mathcal{K}}$ be as defined in (7). Then $\beta_{\mathcal{O}/\mathcal{K}}$ is the only natural transformation such that $\langle id_{\mathit{Sig}_\mathcal{B}}, \alpha_{id}, \beta_{\mathcal{O}/\mathcal{K}} \rangle : \mathit{Inst}_\mathcal{O} \longrightarrow \mathit{Inst}_\mathcal{K}$ is an institution morphism.*

In the other direction, i.e. from $\mathit{Inst}_\mathcal{K}$ to $\mathit{Inst}_\mathcal{O}$, we have to associate ordinal conditional functions to total preorder R. Each such R induces a partitioning $\Omega_0, \Omega_1, \ldots$ of Ω, and we set $\kappa_R(\omega) = i$ iff $\omega \in \Omega_i$. κ_R is an ordinal conditional function, and choosing the natural transformation

$$\beta_{\mathcal{K}/\mathcal{O}} : \mathit{Mod}_\mathcal{K} \Longrightarrow \mathit{Mod}_\mathcal{O}, \beta_{\mathcal{K}/\mathcal{O},\Sigma}(R) = \kappa_{R,\Sigma} \qquad (8)$$

gives rise to an institution morphism.

Proposition 12 *Let $\beta_{\mathcal{K}/\mathcal{O}}$ be as defined in (8). Then $\langle id_{\mathit{Sig}_\mathcal{B}}, \alpha_{id}, \beta_{\mathcal{K}/\mathcal{O}} \rangle : \mathit{Inst}_\mathcal{K} \longrightarrow \mathit{Inst}_\mathcal{O}$ is an institution morphism.*

Note that in this case, $\beta_{\mathcal{K}/\mathcal{O}}$ is not uniquely determined by the satisfaction condition, hence the institution morphism is not unique.

Related and further work

In this paper, we focus on qualitative conditional semantics. One of the most popular semantics of conditionals, however, is the one which is based on conditional probabilities. We proved in (Beierle & Kern-Isberner 2002), that in such a quantitative environment, an institution of probabilistic conditionals $\mathit{Inst}_\mathcal{C}$ can be defined in a very similar way, as was done here in a qualitative framework. But crucial differences became apparent when studying relationships between $\mathit{Inst}_\mathcal{C}$ and $\mathit{Inst}_\mathcal{K}$ via morphisms. As might have been

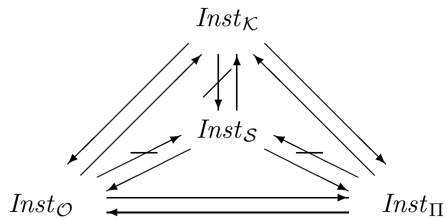

Figure 1: Institution morphisms between $Inst_{\mathcal{K}}$, $Inst_{\mathcal{O}}$, $Inst_{\Pi}$, and $Inst_{\mathcal{S}}$

expected, a unique institution morphism going from $Inst_{\mathcal{C}}$ to $Inst_{\mathcal{K}}$ can be defined in a canonical way, but in principle, there is no morphism going in the other direction. As we found morphisms between $Inst_{\mathcal{O}}$, $Inst_{\Pi}$, and $Inst_{\mathcal{K}}$ in the previous section, similar results hold for all three institutions. This means that the probabilistic semantics for conditionals can be projected onto each qualitative semantics, but there is no way to recover the richness of conditional probabilities from within a qualitative semantical framework.

A particular interesting comparison can be made between the conditional semantics studied here and the qualitative probabilistic semantics defined via so-called *atomic bound systems*, introduced by Snow in (Snow 1994; 1996) and also known as *big-stepped probabilities* (this more intuitive name was coined by Benferhat, Dubois & Prade, see (Benferhat, Dubois, & Prade 1999)). Big-stepped probability distributions P are such that all $P(\omega)$ are positive and form a linearly ordered set, and it holds that for all $\omega_0 \in \Omega$, $P(\omega_0) > \sum_{\omega: P(\omega_0) > P(\omega)} P(\omega)$. So, in a big-stepped probability distribution, the probability of each possible world is bigger than the sum of all probabilities of less probable worlds. Big-stepped probabilities actually provide a standard probabilistic semantics for system P (cf. (Benferhat, Dubois, & Prade 1999)), by accepting conditionals $(B|A)$ iff $P(AB) > P(A\overline{B})$. Again, an institution $Inst_{\mathcal{S}}$ of big-stepped probabilities can be defined (cf. (Beierle & Kern-Isberner 2003)). Although the big-stepped conditional semantics seems to be much more coarse-grained than full probabilistic semantics, and much closer to qualitative conditional semantics, we showed in (Beierle & Kern-Isberner 2003) that there is still a gap between $Inst_{\mathcal{K}}$ and $Inst_{\mathcal{S}}$: There is no institution morphism from $Inst_{\mathcal{K}}$ to $Inst_{\mathcal{S}}$ (but a canonical morphism going into the other direction). Hence, in spite of superficial similarity, basic differences between these two kinds of qualitative reasoning could be made clear by using the formal framework of institutions. Similar results hold for the possibilistic and ordinal semantics, too. Figure 1 illustrates the existing and non-existing links via morphisms between the institutions of purely qualitative, possibilistic, ordinal, and big-stepped conditionals, respectively.

Conclusion

In this paper, we investigated formal logical relationships between various popular semantics for conditionals. By making use of the abstract framework of institutions for specifying logical systems, we were able to prove that the purely qualitative semantics based on plausibility, the ordinal semantics using rankings and the possibilistic semantics are all similar, but each semantics allows specific features of expressiveness. The purely qualitative conditional institution $Inst_{\mathcal{K}}$ provides the weakest semantics, since the models of both the possibilistic and the ordinal institutions can be mapped uniquely to a model of $Inst_{\mathcal{K}}$, but each model of $Inst_{\mathcal{K}}$ gives rise to a class of models of the other institutions.

As part of our future work, we will use institutions for semantics induced by so-called *conditional valuation functions* (Kern-Isberner 2001), aiming at completing quite a general formal picture of conditional semantics.

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