Equivalence of Red-Black and 2-3 Trees

Theorem:
1. Every red-black tree of black height $h$ and $n$ black vertices can be converted to an equivalent 2-3 tree of height $h-1$ and $n$ vertices.
2. Every 2-3 tree of height $h$ and $n$ vertices can be converted to an equivalent red-black tree of black height $h+1$ and $n$ black vertices.

Proof of part 1: Merge each red vertex with its black parent to form a single vertex with two data items (leaf with 0 children or non-leaf with 3 children).

Proof of Part 2:
- Color all vertices black.
- For each leaf with two items, remove the smaller one and put it in a left child colored red.
- For each non-leaf vertex $v$ that contains two items (and has three subtrees), remove the smaller of the two items, create a new red vertex $w$ with this item, remove the left and middle subtrees of $v$ and make them the left and right subtrees of $w$, and make $w$ the left child of $v$.

Note: Construction is not unique; left/right and smaller/larger can be exchanged.
Example: The figure below shows a 2-3 tree on the bottom and three equivalent red-black trees above it.