Viewing in 3D

Foley & Van Dam, Chapter 6
Viewing in 3D

- Transformation Pipeline
- Viewing Plane
- Viewing Coordinate System
- Projections
  - Orthographic
  - Perspective
OpenGL Transformation Pipeline

Homogeneous coordinates in World System

ModelView Matrix

Projection Matrix

Clipping

Viewport Transformation

Window Coordinates
Viewing Coordinate System

Tractor System

Front-Wheel System

Viewer System

Viewing plane

World Coordinate System

$X_w$, $Y_w$, $Z_w$

Viewer Coordinate System

$X_v$, $Y_v$, $Z_v$

$P_0$
Specifying the Viewing Coordinates

- Viewing Coordinates system, \([x_v, y_v, z_v]\), describes 3D objects with respect to a viewer

- A viewing plane (\emph{projection plane}) is set up perpendicular to \(z_v\) and aligned with \((x_v, y_v)\)

- In order to specify a viewing plane we have to specify:
  - a vector \(\mathbf{N}\) normal to the plane
  - a viewing-up vector \(\mathbf{V}\)
  - a point on the viewing plane
Specifying the Viewing Coordinates

- $P_0 = (x_0, y_0, z_0)$ is the point where a camera is located.
- $P$ is a point to look at.
- $N = (P_0 - P)/|P_0 - P|$ is the view-plane normal vector.
- $V = z_w$ is the view up vector, whose projection onto the view-plane is directed up.
Viewing Coordinate System

\[ z_v = N \quad ; \quad x_v = \frac{V \times N}{|V \times N|} \quad ; \quad y_v = z_v \times x_v \]

• The transformation \( M \), from world-coordinate into viewing-coordinates is:

\[
M = \begin{bmatrix}
    x^1_v & x^2_v & x^3_v & 0 \\
    y^1_v & y^2_v & y^3_v & 0 \\
    z^1_v & z^2_v & z^3_v & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & -x_0 \\
    0 & 1 & 0 & -y_0 \\
    0 & 0 & 1 & -z_0 \\
    0 & 0 & 0 & 1
\end{bmatrix} = R \cdot T
\]

• Defining the camera in OpenGL:

```c
glMatrixMode(GL_MODELVIEW);
gluLookAt(P_0x, P_0y, P_0z, P_x, P_y, P_z, V_x, V_y, V_z);
```
Projections

• Viewing 3D objects on a 2D display requires a mapping from 3D to 2D

• A projection is formed by the intersection of certain lines (*projectors*) with the view plane

• Projectors are lines from the *center of projection* through each point in the object
Projections

• Center of projection at infinity results with a parallel projection

• A finite center of projection results with a perspective projection
Projections

• **Parallel projections** preserve relative proportions of objects, but do not give realistic appearance (commonly used in engineering drawing)

• **Perspective projections** produce realistic appearance, but do not preserve relative proportions
Parallel Projection

- Projectors are all parallel
- **Orthographic**: Projectors are perpendicular to the projection plane
- **Oblique**: Projectors are not necessarily perpendicular to the projection plane
Orthographic Projection

Since the viewing plane is aligned with \((x_v, y_v)\), orthographic projection is performed by:

\[
\begin{bmatrix}
    x_p \\
    y_p \\
    0 \\
    1
\end{bmatrix}
= \begin{bmatrix}
    x_v \\
    y_v \\
    0 \\
    1
\end{bmatrix}
= \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x_v \\
    y_v \\
    z_v \\
    1
\end{bmatrix}
\]
Orthographic Projection

- Lengths and angles of faces parallel to the viewing planes are preserved
- **Problem**: 3D nature of projected objects is difficult to deduce
Oblique Projection

• Projectors are *not* perpendicular to the viewing plane
• Angles and lengths are preserved for faces parallel to the plane of projection
• Preserves 3D nature of an object
Oblique Projection

• Two types of oblique projections are commonly used:
  – **Cavalier**: $\alpha = 45^\circ = \tan^{-1}(1)$
  – **Cabinet**: $\alpha = \tan^{-1}(2) \approx 63.4^\circ$
Oblique Projection

\[
\begin{bmatrix}
  x_p \\
  y_p \\
  0 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & a \cos \phi & 0 \\
  0 & 1 & a \sin \phi & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x_v \\
  y_v \\
  z_v \\
  1
\end{bmatrix}
= \begin{bmatrix}
  x_v + z_v a \cos \phi \\
  y_v + z_v a \sin \phi \\
  0 \\
  1
\end{bmatrix}
\]

\[
\frac{1}{a} = \tan(\alpha) \\
z/b = 1/a \\
b = za
\]

\[
x_p = z \cdot a \cdot \cos(\phi) \\
y_p = z \cdot a \cdot \sin(\phi)
\]
Oblique Projection

Cavalier Projections of a cube for two values of angle $\phi$:

- $\phi = 45^\circ$
- $\phi = 30^\circ$

Cabinet Projections of a cube for two values of angle $\phi$:

- $\phi = 45^\circ$
- $\phi = 30^\circ$
Oblique Projection

- **Cavalier** projection:
  - Preserves lengths of lines perpendicular to the viewing plane
  - 3D nature can be captured but shape seems distorted
  - Can display a combination of front, side, and top views

- **Cabinet** projection:
  - Lines perpendicular to the viewing plane project at 1/2 of their length
  - A more realistic view than the Cavalier projection
  - Can display a combination of front, side, and top views
Perspective Projection

- In a perspective projection, the center of projection is at a finite distance from the viewing plane.
- The size of a projected object is inversely proportional to its distance from the viewing plane.
- Parallel lines that are not parallel to the viewing plane converge to a vanishing point.
- A vanishing point is the projection of a point at infinite distance.

![Diagram showing perspective projection and vanishing points](image-url)
Perspective Projection
Vanishing Points

• There are infinitely many general vanishing points.
• There can be up to three *principal vanishing points* (axis vanishing points).
• Perspective projections are categorized by the number of principal vanishing points, equal to the number of principal axes intersected by the viewing plane.
• Most commonly used: one-point and two-points perspective.
Vanishing Points

One point (z axis) perspective projection

Two points perspective projection
Using similar triangles it follows:

\[
\frac{x_p}{d} = \frac{x}{z + d} \quad ; \quad \frac{y_p}{d} = \frac{y}{z + d} \quad ; \\
\frac{x_p}{z + d} = \frac{d \cdot x}{z + d} \quad ; \quad \frac{y_p}{z + d} = \frac{d \cdot y}{z + d} \quad ; \quad z_p = 0
\]
Thus, a perspective projection matrix is defined as:

$$
M_{\text{per}} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{d} & 1
\end{bmatrix}
$$

$$
M_{\text{per}} P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{d} & 1
\end{bmatrix}\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
0 \\
\frac{z + d}{d}
\end{bmatrix}
$$

$$
x_p = \frac{d \cdot x}{z + d} \quad ; \quad y_p = \frac{d \cdot y}{z + d} \quad ; \quad z_p = 0
$$
Perspective Projection

• $M_{\text{per}}$ is singular ($|M_{\text{per}}|=0$), thus $M_{\text{per}}$ is a many to one mapping (for example: $M_{\text{per}}P=M_{\text{per}}2P$)

• Points on the viewing plane ($z=0$) do not change

• The homogeneous coordinates of a point at infinity directed to $(U_x,U_y,U_z)$ are $(U_x,U_y,U_z,0)$. Thus, The vanishing point of parallel lines directed to $(U_x,U_y,U_z)$ is at $[dU_x/U_z, dU_y/U_z]$  

• When $d \to \infty$, $M_{\text{per}} \to M_{\text{ort}}$
Projections

What is the difference between moving the center of projection and moving the projection plane?

Original

Moving the Center of Projection

Moving the Projection Plane