Representing Curves – Part II

Foley & Van Dam, Chapter 11
Representing Curves

• Polynomial Splines
  • Bezier Curves
  • Cardinal Splines
  • Uniform, non rational B-Splines
• Drawing Curves
• Applications of Bezier splines
Bezier Curves

• Specify indirectly the tangent with two points that are not on the curve
Beziers Curves

- Easy to enforce $C^0$ and $C^1$ continuity. If $P_{j-1}$, $P_j$, and $P_{j+1}$ are collinear then the curve is $G^1$ in $P_j$
- Similar to Cardinal Splines
Beziers Curves

- No need to supply tangents
- For each segment curve between $P_{k-1}$ and $P_{k+2}$, we have:
  - $V(0) = P_{k-1}$
  - $V(1) = P_{k+2}$
  - $V'(0) = 3(P_k - P_{k-1})$
  - $V'(1) = 3(P_{k+2} - P_{k+1})$
Beziers Curves

- The relation between the Hermite geometry vector and the Bezier geometry vector is:

\[
G_H = \begin{bmatrix}
    p_k \\
p_{k+1} \\
T_k \\
T_{k+1}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    -3 & 3 & 0 & 0 \\
    0 & 0 & -3 & 3
\end{bmatrix} \begin{bmatrix}
    p_{k-1} \\
p_k \\
p_{k+1} \\
p_{k+2}
\end{bmatrix} = M_{BH} G_B
\]

- Combining with the Hermite interpolation:

\[
V(u) = U(u) M_H M_{BH} G_B = \begin{bmatrix}
    u^3 \\
    u^2 \\
    u \\
    1
\end{bmatrix}^T \begin{bmatrix}
    -1 & 3 & -3 & 1 \\
    3 & -6 & 3 & 0 \\
    -3 & 3 & 0 & 0 \\
    1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    p_{k-1} \\
p_k \\
p_{k+1} \\
p_{k+2}
\end{bmatrix}
\]
**Bezier Curves**

\[ V(u) = U(u) M_H M_{BH} G_B = \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} \]

Blending functions

\[ U(u)M_HM_{BH} : \text{Bernstein polynomials:} \]

\[ B_i^n(u) = \binom{n}{i} u^i (1 - u)^{n-i} \]

with \[ \binom{n}{i} = \frac{n!}{(n-i)!i!} \]

Note: Assume values between 0 and 1
Bezier Curves

- Bezier curves produces $C^1$ continuous curves
- Linear (convex) combination of 4 basis functions. Alternatively, it is a convex combination of 4 control points

**Advantage:**
- No need for tangents
- Curve always contained in the convex hull

**Disadvantage:**
- Tangent approximation might be imprecise

**NOTE:** The constant 3 is obtained by assuming that

$$V'(0) = \beta (P_k - P_{k-1}) \quad \text{and} \quad V'(1) = \beta (P_{k+2} - P_{k+1})$$

Then deriving $\beta$ such that the Bezier curve between

$P_{k-1} = (0,0), \ P_k = (1,0), \ P_{k+1} = (2,0), \ P_{k+2} = (3,0)$

has constant velocity between $P_{k-1}$ and $P_{k+2}$
Cardinal Splines

- No need to supply tangents
- For each segment curve between $P_k$ and $P_{k+1}$, we have:
  - $V(0) = P_k$
  - $V(1) = P_{k+1}$
  - $V'(0) = s(P_{k+1} - P_{k-1})$ (s = Tension Parameter)
  - $V'(1) = s(P_{k+2} - P_k)$
Cardinal Splines

• The relation between the Hermite geometry vector and the Cardinal geometry vector is:

\[
G_H = \begin{bmatrix}
  p_k \\
  p_{k+1} \\
  T_k \\
  T_{k+1}
\end{bmatrix}
= \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -s & 0 & s & 0 \\
  0 & -s & 0 & s
\end{bmatrix}
\begin{bmatrix}
  p_{k-1} \\
  p_k \\
  p_{k+1} \\
  p_{k+2}
\end{bmatrix}
= M_{HC} G_C
\]

• Combining with the Hermite interpolation:

\[
V(u) = U(u) M_H M_{CH} G_C = \begin{bmatrix}
  u^3 \\
  u^2 \\
  u \\
  1
\end{bmatrix}^T \begin{bmatrix}
  -s & 2-s & s-2 & s \\
  2s & s-3 & 3-2s & -s \\
  -s & 0 & s & 0 \\
  0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  p_{k-1} \\
  p_k \\
  p_{k+1} \\
  p_{k+2}
\end{bmatrix}
\]
Cardinal Splines

Cardinal spline blending functions for $s=1$

Cardinal spline blending functions for $s=2$

\[ V(u) = C_0(u)P_{k-1} + C_1(u)P_k + C_2(u)P_{k+1} + C_3(u)P_{k+2} \]
Cardinal Splines

- Cardinal spline produces $C^1$ continuous curve
- Linear combination of 4 basis functions. Alternatively, it is a linear combination of 4 control points
- **Advantage:** No need for tangents
- **Disadvantage:** Tangent approximation might be imprecise

![Small tension](image1.png) ![Big tension](image2.png)
B-Spline Approximation

- A cubic B-spline (Basis Spline) approximates \( m \geq 3 \) points \( P_0, P_1, \ldots, P_m \) with a curve consisting of \( m-2 \) cubic polynomial curve segments \( Q_3, Q_4, \ldots, Q_m \).

- Segment \( Q_i(t) \) is defined for \( t \in [0,1] \), but with the variable substitution \( t = t+k \) we can make the domains of the segments sequential so that \( Q_i(t) \) is defined for \( t_i \leq t < t_{i+1} \).

- The values of the curve for \( t_i, i \geq 3 \) are called knots (there are \( m-1 \) knots).

- The segment \( Q_i, i \geq 3 \) is defined by 4 control points: \( P_{i-3}, P_{i-2}, P_{i-1}, P_i \).
B-Spline Approximation

For Hermite and Bezier curves we have:

\[ Q(t) = T \cdot M \cdot G, \quad t \in [0,1] \]

Define

\[ T_i = [(t-t_i)^3, (t-t_i)^2, (t-t_i), 1] \]

The B-spline formulation for the segment \( Q_i(t) \) is:

\[ Q_i(t) = T_i \cdot M_{Bs} \cdot G_{Bs} \]

\[ = [(t-t_i)^3, (t-t_i)^2, (t-t_i), 1] \cdot \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix} \]
B-Spline Approximation

• B-splines are **uniform** when the knots are spaced at equal intervals of $t$

• Uniform B-splines use the same blending function for each $Q_i(t)$

• **Non rational** when $x(t)$, $y(t)$ and $z(t)$ are not defined as ratio between two polynomial
B-Spline Approximation

- **Properties:**
  - Changing the control points has a **local** effect
  - B-splines are $C^0$, $C^1$ and $C^2$ continuous
  - The curve is contained in the convex hull defined by the control points
  - The curve can be closed by repeating the first three control points: $P_0, P_1, P_2, \ldots, P_m, P_0, P_1, P_2$
  - Little control on where the spline goes (drawback)
Drawing Curves

- Direct evaluation of the parametric polynomial (Horner’s rule)
- Forward finite differences (explained in class)
- Recursive subdivision (stop dividing and draw a line when segment is flat)
Applications of Bezier splines

• A quadratic Bezier curve is defined in terms of three control points:

\[ V(u) = U(u)MG = \begin{bmatrix} u^2 \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \end{bmatrix} \]

• It is always possible to represent a quadratic Bezier curve with a cubic Bezier curve (just set to zero \( u^3 \) coefficient)

• The inverse is not true: most cubic curves cannot be represented exactly by quadratic curves. Sometimes, not even by series of quadratic curves.
Applications of Bezier splines

• Outlines of Postscript and TrueType characters are defined in terms of Bezier curves
• Postscript uses cubic forms and TrueType uses quadratic forms
• Converting TrueType to Postscript is trivial; the opposite can be done only with approximations

Control points defining the outline of the letter 'b' of Monotype Arial. On-curve points are indicated with a square and off-curve points with crosses.
Applications of Bezier splines

- This representation can be scaled to arbitrary sizes, rotated, etc..
- Finally the outline is rasterized and filled
- The final quality can be improved with antialiasing
- When the font size is small, “hinting” may be necessary to improve symmetry and readability

Before hinting  After hinting