Representing Curves – Part II

Foley & Van Dam, Chapter 11

Representing Curves

- Polynomial Splines
- Bezier Curves
- Cardinal Splines
- Uniform, non rational B-Splines
- Drawing Curves
- Applications of Bezier splines

Bezier Curves

- Specify indirectly the tangent with two points that are not on the curve

- Easy to enforce $C^0$ and $C^1$ continuity. If $P_{j-1}$, $P_j$ and $P_{j+1}$ are collinear then the curve is $G^1$ in $P_j$
  - Similar to Cardinal Splines

- No need to supply tangents
  - For each segment curve between $P_{k-1}$ and $P_{k+2}$, we have:
    - $V(0)=P_{k-1}$
    - $V(1)=P_{k+2}$
    - $V'(0)=3(P_k-P_{k-1})$
    - $V'(1)=3(P_{k+2}-P_{k+1})$

- The relation between the Hermite geometry vector and the Bezier geometry vector is:

$$G_h = \begin{bmatrix} p_0 \\ p_{k-1} \\ u_0 \\ T_k \\ T_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} = M_{h,h} G_h$$

- Combining with the Hermite interpolation:

$$V(u) = U(u)M_{h,h} M_{g,h} G_g = \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$
Bezier Curves

\[
V(u) = U(u) M_{uu} M_{uv} G_u = \begin{bmatrix}
u^3 & u^2 & u & 1
\end{bmatrix} \begin{bmatrix}
-1 & 3 & -3 & 1
3 & -6 & 3 & 0
-3 & 0 & 0 & 0
1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
p_{k-1}
p_k
p_{k+1}
p_{k+2}
\end{bmatrix}
\]

Blending functions

\[U(u)|_{u=M_{uu}} : \text{Bernstein polynomials:}
B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}
\]

with
\[
\frac{n!}{(n-i)! i!}
\]

Note: Assume values between 0 and 1

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Cardinal Splines

- No need to supply tangents
- For each segment curve between \(P_k\) and \(P_{k+1}\), we have:
  - \(V(0)=P_k\)
  - \(V(1)=P_{k+1}\)
  - \(V(0)=s(P_{k+1}-P_{k-1})\) (s = Tension Parameter)
  - \(V(1)=s(P_{k+2}-P_k)\)

\[\text{Cardinal spline blending functions for } s=1\]

\[\text{Cardinal spline blending functions for } s=2\]

- The relation between the Hermite geometry vector and the Cardinal geometry vector is:
\[
G_u = \begin{bmatrix}
p_k
p_{k+1} - s P_{k-1}
s P_{k-1}
0
\end{bmatrix} = M_{uu} G_c
\]

- Combining with the Hermite interpolation:
\[
V(u) = U(u) M_{uu} M_{uv} G_u = \begin{bmatrix}
\frac{u^3}{3!} & \frac{u^2}{2!} & \frac{u}{1!}
\end{bmatrix} \begin{bmatrix}
-1 & 3 & -3 & 1
3 & -6 & 3 & 0
-3 & 0 & 0 & 0
1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
p_{k-1}
p_k
p_{k+1}
p_{k+2}
\end{bmatrix}
\]

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Cardinal Splines

- Cardinal spline produces \(C^1\) continuous curve
- Linear combination of 4 basis functions. Alternatively, it is a linear combination of 4 control points
- **Advantage:** No need for tangents
- **Disadvantage:** Tangent approximation might be imprecise

Small tension

Big tension
B-Spline Approximation

- A cubic B-spline (Basis Spline) approximates \( m \geq 3 \) points \( P_0, P_1, \ldots, P_m \) with a curve consisting of \( m-2 \) cubic polynomial curve segments \( Q_3, Q_4, \ldots, Q_m \).

- Segment \( Q_i(t) \) is defined for \( t \in [0,1] \), but with the variable substitution \( t = t + k \) we can make the domains of the segments sequential so that \( Q_i(t) \) is defined for \( t_i < t_{i+1} \).

- The values of the curve for \( t_i, i \geq 3 \) are called knots (there are \( m-1 \) knots).

- The segment \( Q_i, i \geq 3 \) is defined by 4 control points: \( p_{i-3}, p_{i-2}, p_{i-1}, p_i \).

B-Spline Approximation

For Hermite and Bezier curves we have:

\[
Q(t) = T \cdot M \cdot G, \quad t \in [0,1]
\]

Define

\[
T_i = [(t-t_i)^3, (t-t_i)^2, (t-t_i), 1]
\]

The B-spline formulation for the segment \( Q_i(t) \) is:

\[
Q_i(t) = T_i \cdot M_{Bs} \cdot G_{Bs}
\]

B-Spline Approximation

- Properties:
  - Changing the control points has a local effect.
  - B-splines are \( C^0, C^1 \) and \( C^2 \) continuous.
  - The curve is contained in the convex hull defined by the control points.
  - The curve can be closed by repeating the first three control points: \( P_0, P_1, P_2, \ldots, P_m, P_0, P_1, P_2 \).
  - Little control on where the spline goes (drawback).

Drawing Curves

- Direct evaluation of the parametric polynomial (Horner’s rule).
- Forward finite differences (explained in class).
- Recursive subdivision (stop dividing and draw a line when segment is flat).

Applications of Bezier splines

- A quadratic Bezier curve is defined in terms of three control points:

\[
V(u) = U(u) \cdot M \cdot G = \begin{bmatrix} u^2 & 1 \\ \end{bmatrix} \begin{bmatrix} 1 & \end{bmatrix} \begin{bmatrix} P_{-1} \\ P_0 \\ P_1 \\ P_2 \end{bmatrix}
\]

- It is always possible to represent a quadratic Bezier curve with a cubic Bezier curve (just set to zero \( u^3 \) coefficient).
- The inverse is not true: most cubic curves cannot be represented exactly by quadratic curves.

Sometimes, not even by series of quadratic curves.
Applications of Bezier splines

- Outlines of Postscript and TrueType characters are defined in terms of Bezier curves
- Postscript uses cubic forms and TrueType uses quadratic forms
- Converting TrueType to Postscript is trivial; the opposite can be done only with approximations

Control points defining the outline of the letter ‘b’ of Monotype Arial. On-curve points are indicated with a square and off-curve points with crosses.

Applications of Bezier splines

- This representation can be scaled to arbitrary sizes, rotated, etc.
- Finally the outline is rasterized and filled
- The final quality can be improved with antialiasing
- When the font size is small, “hinting” may be necessary to improve symmetry and readability

Before hinting

After hinting