Mathematics of Domains

A Dissertation

Presented to

The Faculty of the Graduate School of Arts and Sciences

Brandeis University

Department of Computer Science

Harry G. Mairson, Advisor

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by

Michael A. Bukatin

February 2002

This dissertation, directed and approved by Michael A. Bukatin's Committee, has been accepted and approved by the Graduate Faculty of Brandeis University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Dean of Arts and Sciences

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Dedicated to the memory of Anya Pogosyants and Igor Slobodkin

ACKNOWLEDGMENTS

In chronological order, I would like to thank Alexander Shen', who introduced me to domain theory, which he used in his remarkable PhD Thesis on algoritmic variants of the notion of entropy, and Alexander Saevsky, for fruitful collaboration in 1986-1989, which enabled my results on subdomains and finitary retractions, presented here.

I would like to thank my advisor Harry Mairson for arranging financial support from Fall 1992 to Spring 1996 under NSF Grant CCR-9216185 and Office of Naval Research Grant ONR N00014-93-1-1015 and for numerous fruitful discussions.

I am especially thankful to Joshua Scott and Svetlana Shorina, for the joint work in my research project in the field of analysis on domains. The results of our joint papers are the most significant results reported in this Thesis.

The influence of Abbas Edalat, Bob Flagg, and Klaus Keimel was quite crucial for some parts of this Thesis.

I would also like to acknowledge helpful communications with Michael Alekhnovich, Alexander Artemyev, Will Clinger, Yuri Ershov, Nadezhda Fradkova, Carl Gunter, Reinhold Heckmann, Michael Huth, Ataliya Kagan, Ralph Kopperman, Elena Machkasova, Yuri Matiyasevich, Steve Matthews, Albert Meyer, Michael Mislove, Robert Muller, Simon O'Neill, Gordon Plotkin, Jeffrey Sanchez, Vladimir Sazonov, Ross Viselman, Steve Vickers, and Mitch Wand.

ABSTRACT

Mathematics of Domains

A dissertation presented to the Faculty of the Graduate School of Arts and Sciences of Brandeis University, Waltham, Massachusetts

by Michael A. Bukatin

Two groups of naturally arising questions in the mathematical theory of *domains for denotational semantics* are addressed. Domains are equipped with *Scott topology* and represent *data types*. *Scott continuous functions* represent *computable functions* and form the most popular continuous model of computations.

Covariant Logic of Domains: Domains are represented as *sets of theories*, and Scott continuous functions are represented as *input-output inference engines*. The questions addressed are: **A.** What constitutes a *subdomain*? Do subdomains of a given domain A form a domain? **B.** Which *retractions* are *finitary*? **C.** What is the essence of generalizations of *information systems* based on *non-reflexive logics*? Are these generalizations restricted to continuous domains?

Analysis on Domains:

D. How to describe Scott topologies via **generalized distance functions** satisfying the requirement of Scott continuity ("abstract computability")? The answer is that the axiom $\rho(x, x) = 0$ is incompatible with Scott continuity of distance functions. The resulting **relaxed metrics** are studied.

E. Is it possible to obtain Scott continuous relaxed metrics via *measures* of domain subsets representing *positive* and *negative information* about domain elements? The positive answer is obtained via the discovery of the novel class of **co-continuous valuations** on the systems of Scott open sets.

Some of these natural questions were studied earlier. However, in each case a novel approach is presented, and the answers are supplied with much more compelling and clear justifications, than were known before.

Chapter 0

Preface

This text represents the results of my research in mathematics of approximation domains equipped with Scott topology. I would be the first to classify this research as belonging to the field of pure mathematics, yet it is being presented as a dissertation in computer science. I think, this demands some explanation.

0.1 Accepted Practice

There is a widely recognized body of computer science applications of the mathematical theory of domains, reflected in dozens of monographs, and hundreds, if not thousands, of research papers (some of the related references are in the text). Some of the most prominent contributors to this theory work at departments of computer science, rather than at departments of mathematics. Due to these reasons, it is quite often that purely mathematical research in this field is presented as doctoral dissertations in computer science, even in the leading schools.

For example, a famous work by Kim Wagner [57] defended as a PhD Thesis in the School of Computer Science of Carnegie Mellon University, clearly belongs to this class of research. As my main achievements to this moment are in mathematical theory, and not in its applications, I believe it appropriate to follow this practice.

0.2 Recongnized Applications

The main recognized application of domains is to serve as denotational models of programming languages. This application is due to Christofer Strachey, who introduced the idea of denotational semantics, and Dana Scott, who invented approximation domains equipped with what is now called Scott topology and constructed extensional models of lambda-calculus by solving reflexive domain equations like $D \cong [D \to D]$. The classical textbook describing this class of applications is [53], and there are many successors. It is widely argued that denotational descriptions of programming languages using approximation domains should be considered canonical definitions of those languages, while other types of descriptions should be considered as derivative. This viewpoint obviously has its share of opponents too.

Among more hands-on applications of this approach are verifications of compiler correctness, uses of abstract interpretation to perform some static analysis of programs, attempts to produce logical frameworks based on denotational semantics, etc.

Another important application, which emerged in the nineties, is to use domains to produce computational models for classical mathematical structures and thus to provide the framework for computations in such structures. This direction of research was originated by Abbas Edalat, who used domains to generalize the notion of Riemann integral to fractal spaces and obtained more efficient methods to compute integrals on such spaces (see [19] and references therein). The research group at Imperial College headed by Edalat is now working towards constructing systematic models of this kind for a wide class of mathematical structures (see [20] for the review of these efforts).

0.3 Potential Applications

Many researchers in this field probably have their own lists of prospective applications, which often do not reduce to the two classes of applications described before. My own list includes the following.

When denotational semantics was first introduced, it was suggested that the canonical approach would be not to produce denotational description of existing languages, but to design languages fitting the denotational models of interest. While this approach has not received sufficient development, I believe that the potential for such development exists.

Another class of applications is related to the potentially possible development of our abilities to perform computations in domains with reasonable efficiency, i.e. to be able to find reasonable approximations for a sufficiently wide class of definitions of domain elements and Scott continuous functions while spending realistic amounts of resources. We call this open problem **Problem A** from now on in this text. I cannot predict at this point, whether such a development is actually possible. However, if it is possible, one immediate consequence would be the possibility of a generator of prototype implementations of programming languages based on their denotational semantics.

The central part of this monograph develops theories of relaxed metrics and co-continuous valuations on domains. This development is just one manifestation of the rapid evolution of analysis on domains taking place recently. There can be little doubt that a lot of analogs of basic structures of functional analysis will soon emerge for domains. If Problem A is actually solved, these analogs should enable us to transfer various methods of optimization known in classical mathematics into the computational setting and should lead to the development of the whole new class of advanced methods for machine learning.

Thus Problem A is quite crucial for the further applicability of this field.

0.4 Simplicity of Presentation and Prerequsites

I make an emphasis on simplicity and intuitive clarity of presentation. The techniques here are deliberately elementary and non-categorical, except for those rare cases when categorical methods are unavoidable.

The Introduction can be used as a brief tutorial for the readers, who are not familiar with the field. The Introduction sacrifices the completeness of presentation of the standard information on domains for the accessibility.

Thus the knowledge of domain theory and category theory is not strictly required to read this text. The reader is, however, expected to be somewhat familiar with basic notions of the following fields: formal logic, theory of metric and topological spaces, measure theory.

0.5 Structure of the Text

The Introduction can be thought of as a short tutorial in domain theory and a repository of its basic definitions. It also contains a detailed overview of other parts of the text.

Part II and Part III constitute the core of the Thesis. Part II presents our results related to the logic of spaces of fixed points. Part III presents our results in the theory of generalized distances and measures for domains.

0.6 Electronic Coordinates

Here are some relevant electronic coordinates. I hope they will be alive for a while.

My e-mail address: `bukatin@@cs.brandeis.edu`

This thesis: `http://www.cs.brandeis.edu/~bukatin/thesis.ps.gz`

My papers in computer science:

`http://www.cs.brandeis.edu/~bukatin/papers.html`

Contents

Part I

Introduction

Chapter 1

Continuity in Computations and Scott Domains

Here we present evidence of the continuous nature of computations [10].

1.1 Continuity in Software Engineering

It seems appropriate to start with a real life observation before dealing with mathematical models. Virtually all software products used today contain a number of bugs, which lead to refusal to work, crashes, and incorrect processing of data. Yet for a usable software product these unpleasant effects are observed relatively infrequently, as most of the time the product works satisfactory. What we should say here is that the real product is sufficiently close to the ideal product, close enough to be used in lieu of the unavailable ideal product. This is a phenomenon of continuity. This continuity results from the process of testing and amending the software and is crucial for our ability to use any software at all. Moreover, it is quite possible that the ideal product cannot exist at all (e.g. if the specifications are for an undecidable problem), yet we still can consider models which contain representatives for such ideal products and for their real-life approximations.

The author so far has not encountered studies dealing with this phenomenon mathematically. Software reliability models do not seem to qualify, because they always abstract from the nature of the particular software involved and just study the dynamics of error rates. We would like to see the difference between the ideal software and its real-life approximation expressed semantically, error classification and rates being built on top of such semantical structure. We think that such approach to software design is potentially more promising than the concept of fully verified software.

1.2 Continuity in Constructive Mathematics

Another evidence of continuity of computations comes from effective versions of mathematical analysis (constructive, recursive, or intuitionistic analysis — see, for example [6, 25]). All constructive functions from constructive real numbers to constructive real numbers must be continuous. It is impossible to properly treat computations at the points of discontinuity. If one has to deal with non-continuous functions, they have to be partially defined — points of discontinuity cannot belong to their domain (although see [18]).

1.3 Continuity in Denotational Semantics of Programming Languages

The most important and mathematically rich example of a continuous approach to computation is denotational semantics (see, for example, [44, 46, 48, 53, 59, 2]). We briefly introduce its main ideas here.

Denotational semantics of a formal language is a map S from the space of formal texts to the certain space of meanings (semantic space). $S[\![P]\!]$ is a meaning of the formal text P; equivalently we say that P denotes $S[\![P]\!]$.

To provide denotational semantics of the λ-calculus it was necessary to solve equations $D \cong [D \to D]$ for semantic spaces — the task which looked quite non-trivial and which does not have non-trivial solutions if we interpret $[D \to D]$ as the set of all functions from D to D. Topological ideas gave the solution [45, 53].

Continuity in computations always relates closely to the idea of formal approximation. Interval numbers formally approximate one another, partially defined functions do the same, etc. The relation of formal approximation usually is a partial order, traditionally denoted as $\sqsubseteq$. Properties vary in different approaches, but the existence of the least defined element and of lowest upper bounds of increasing chains (limits of sequences of better and better formal approximations) is usually required. Topology then is introduced in such a way, that the continuous functions are monotonic functions preserving limits of increasing chains. The resulting topological spaces are called *domains*.

It is natural to require of correctly defined computational processes that the better formal approximation of an input is known, the better formal approximation of the corresponding output should be computed (monotonicity). Also well defined computational schemes (e.g. in numerical methods) tend to preserve limits of formal approximations — we can get the result with an arbitrarily small error if we take the input data with sufficient precision and iterate long enough.

Thus continuous functions above are good candidates to serve as models of computable functions. The main paradigm of the domain theory is: "continuous" is the right mathematical model for "computable". The resulting topological spaces possess a number of remarkable properties. Especially important for computer science applications are theorems of fixed points for functions and functors. The existence of canonical fixed points of continuous functions justifies all recursive definitions of functions. The existence of canonical fixed points of continuous functors justifies all recursive definitions of types; in particular, they give the solution of the equation $D \cong [D \to D]$, which is interpreted as homeomorphism, and $[D \to D]$ — as the space of all continuous functions from D to D in the pointwise topology.

Chapter 2

Overview of Results

2.1 Overview of Part II

The elementary covariant logical approach to domains is known under the name of *information systems*. Under this approach domain elements are thought of as *theories* in a logical calculus, and continuous functions are thought of as *inference engines*, deducing information about $f(x)$ from information about x.

This approach provides very simple and powerful intuition about domains, thus making it easier to learn domain theory and to find and motivate new notions and constructions. Thus, it can be recommended as both didactic and research framework.

A more complicated and less elementary contravariant logical approach to domains uses the ideas of *Stone duality*. We use Stone duality only as an auxiliary vehicle in this text.

In Chapter 5 of Part I we introduce the framework of information systems for algebraic Scott domains. We reformulate the notion of consistency in a more technically convenient way compared to the standard framework [48].

Part II is structured as follows. In Chapter 6 we explain the intuition behind the generalization of this approach to the case of continuous Scott domains by R.Hoofman [28]. Based on this intuition we further generalize this approach to domains of fixed points of Scott continuous transformations of powersets. We also discuss the significance of the algebraic case from the logical point of view — algebraic Scott domains correspond to the standard logic, while some of the more general classes of domains result from either Hoofman's non-reflexive logic or Sazonov's non-finitary logic.

The subsequent chapters, which represent our earlier results, may be also viewed as applications of Chapter 6. Chapter 7 studies the notion of *subdomain* and the *domain of subdomains* for the algebraic Scott domains. We show that the closure operations play an exclusive role in the formation of subdomains, as opposed to finitary retractions or projections, and that this exclusive role can be explained by the intuition presented in Chapter 6.

Chapter 8 studies finitary retractions and provides a novel simple criterion for finitarity. Again, the results can be viewed as an illustration for the intuition of Chapter 6. Some of the results of Chapters 7 and 8 might be known as folklore, as Carl Gunter suggested to me, since it is quite likely that somebody might have discovered them independently. However, I was not able to trace any written evidence of that.

2.2 Overview of Part III

This part contains the core results of this Thesis.

Our research program is to develop analogs of classical mathematical structures (metrics, measures, series, vector spaces, etc.) for domains with Scott topology, with hope to be able eventually to use methods of classical continuous mathematics over the spaces of programs.

Our research program is essentially dual to the research program initiated by Abbas Edalat, who suggested to use domains in order to produce computational models for classical mathematical structures and thus provide the framework for computations in such structures [20]. We should point out that these dual research programs share a lot of technical material.

Chapter 9 studies measure theory on domains. This chapter introduces novel classes of *co-continuous valuations* and CC-valuations and builds such valuations for a large class of domains.

Chapter 10 studies theory of generalized distances on domains. We discover that if we would like our generalized distances to be Scott continuous functions and simultaneously to describe Scott topology via these distances, then the axiom $\rho(x, x) = 0$ cannot hold. Accordingly, we introduce a new class of generalized distances with values in interval numbers. We call these distances *relaxed metrics* and build them here for some domains.

Chapter 11 introduces a general mechanism of building relaxed metrics from measure-like structures, and shows how to build relaxed metrics from CC-valuations for continuous Scott domains.

Chapter 12 generalizes the results of the previous chapter to continuous dcpo's by solving the patologies of the behavior of negative information for non-weakly Hausdorff spaces. The connection between negative information and tolerances is explored.

Chapter 3

Main Definitions of Domain Theory

This chapter is rather terse. Its main purpose is to serve as a repository of the main definitions in this text. The two subsequent chapters give important examples of the notions given here and provide necessary intuition.

We assume that the reader is familiar with the notions of partially ordered set (partial order), topological space, and continuous function. In some rare cases we will use simple categorical notions, but the reader is able to skip them without excessive harm to the overall comprehension.

3.1 Notation

We use $U \subset V$ as an equivalent of $U \subseteq V$ & $U \neq V$.

3.2 Domains

3.2.1 Directed Sets

A partially ordered set (poset), $(S, \sqsubseteq)$, is *directed* if $\forall x, y \in S.\ \exists z \in S.\ x \sqsubseteq z, y \sqsubseteq z$. In particular, empty sets are directed.

The notion of directed set should be considered as a generalization of the notion of increasing sequence, $s_1 \sqsubseteq s_2 \sqsubseteq \ldots$.

3.2.2 Directed Complete Partial Orders (DCPO's)

A poset, $(A, \sqsubseteq)$, is a *directed complete partial order* or *dcpo* if for any directed $S \subseteq A$, the least upper bound $\sqcup S$ of S exists in A.

Note that, since the empty set is directed, any dcpo A must have the least element, which we denote as $\perp$ or $\perp_A$.

Because domains without least elements are also considered lately, differences in terminology occur at this point. Sometimes what we call *directed complete partial orders* is called *complete partial orders*, and a larger class of partial orders, where only non-empty directed sets must have least upper bounds and, thus, the least element does not have to exist, is called *directed complete partial orders*.

3.2.3 The "Way Below" Relation and Compact (Finite) Elements

Consider a dcpo $(A, \sqsubseteq)$ and $x, y \in A$. We say that $x \ll y$ (x is *way below* y) if for any directed set $S \subseteq A$, $y \sqsubseteq \sqcup S \Rightarrow \exists s \in S.\ x \sqsubseteq s$.

An element x, such that $x \ll x$, is called *compact* or *finite*.

It is easy to see, that $x \ll y$ implies $x \sqsubseteq y$, however the inverse relationship is, in general, much more complicated. For example, it is possible that $x \ll x$, but it is also possible that $x \sqsubset y$ and $x \not\ll y$ at the same time. A kind of transitivity, $x' \sqsubseteq x, x \ll y, y \sqsubseteq y' \Rightarrow x' \ll y'$, does hold.

3.2.4 Continuous and Algebraic DCPO's

Consider a dcpo $(A, \sqsubseteq)$ and $K \subseteq A$. We say that a dcpo A is a *continuous dcpo* with *basis* K, if for any $a \in A$, the set $K_a = \{k \in K \mid k \ll a\}$ is directed and $a = \sqcup K_a$. We call elements of K *basic* elements.

If a continuous dcpo has a basis consisting only of compact elements, such a dcpo is called *algebraic*.

3.2.5 Bounded Completeness and Domains

We say that A is *bounded complete* if $\forall B \subseteq A.\ (\exists a \in A.\ \forall b \in B. b \sqsubseteq a) \Rightarrow \sqcup_A B$ exists.

If an algebraic dcpo is bounded complete, it is called an *algebraic Scott domain* or, simply, a *Scott domain.*

If a continuous dcpo is bounded complete, it is called a *continuous Scott domain* [28].

To be consistent with these definitions, if a dcpo is bounded complete, we call it a *complete Scott domain.*

3.2.6 Lattices

If each subset of a partially ordered set A has the least upper bound in A, A is called a *complete lattice.*

A complete lattice which is a continuous dcpo is called a *continuous lattice.*

A complete lattice which is an algebraic dcpo is called an *algebraic lattice.*

Note that complete lattices are precisely those complete Scott domains, which have the top element, $\top$. Also note that any subset of a complete Scott domain A also has the greatest lower bound in A.

3.3 Scott Topology and Scott Continuous Functions

3.3.1 Aleksandrov and Scott Topologies

Consider dcpo $(A, \sqsubseteq_A)$ and $U \subseteq A$. U is *Aleksandrov open* if $\forall x, y \in A.\ x \in U, x \sqsubseteq y \Rightarrow y \in U$. An Aleksandrov open set U is *Scott open* if for any directed poset $S \subseteq A$, $\sqcup S \in U \Rightarrow \exists s \in S.\ s \in U$.

It is easy to see that Aleksandrov open sets and Scott open sets form topologies which are called, respectively, the *Aleksandrov topology* and the *Scott topology*.

3.3.2 Continuous functions

Consider dcpo's $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ with the respective Aleksandrov topologies. It it easy to see that a function $f : A \to B$ is (Aleksandrov) continuous iff it is monotonic, i.e. $x \sqsubseteq_A y \Rightarrow f(x) \sqsubseteq_B f(y)$.

Consider dcpo's $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ with the respective Scott topologies. It it easy to see that a function $f : A \to B$ is (Scott) continuous iff it is Aleksandrov continuous and for any directed poset $S \subseteq A$, $f(\sqcup_A S) = \sqcup_B \{f(s) \mid s \in S\}$.

3.4 Functional Spaces

Consider dcpo's A and B. We define the functional space $[A \to B]$ as the set of all Scott continuous functions $f : A \to B$ with the partial order $f \sqsubseteq g \Leftrightarrow \forall x \in A.\ f(x) \sqsubseteq_B g(x)$. It is easy to check that $[A \to B]$ is a dcpo.

3.5 Retractions, Projections, And Their Pairs

Consider dcpo A and its Scott continuous transformation, $f : A \to A$. If $f = f \circ f$, then f is called a *retraction* of A. If f is such a retraction and $f \sqsubseteq id_A$, then f is called a *projection* of A (id_A is the identity transformation $x \mapsto x$ of A).

Consider dcpo's A and B and a pair of Scott continuous functions, $i : B \to A$ and $j : A \to B$, such that $j \circ i = id_B$. Then it is easy to check that $r = i \circ j$ is a retraction of A. In such a situation we call i an *embedding* of B **into** A, j a *retraction* of A **onto** B, and $\langle i, j \rangle$ an *embedding-retraction pair* (or simply, a *retraction pair*). Sometimes, the whole pair is called a retraction of A onto B.

If r is also a projection, then the terms *projection onto*, and *embedding-projection pair* (or simply, *projection pair*) are used in the corresponding situations.

Chapter 4

Interval Numbers

In this chapter we will look at several important domains, including interval numbers, which form the basis for correct approximate computations with real numbers.

4.1 Vertical Segments and Rays of Real Line

All domains considered in this section are linearly ordered. We call all linearly ordered domains *vertical domains.*

A vertical segment of a real line, $[A, B], A < B$, with its natural order, $\sqsubseteq = \leq$, is a continuous lattice. So is a vertical ray $[A, +\infty]$ and the whole real line $[-\infty, +\infty]$ with the same order.

Note that the last two spaces must contain $+\infty$ for directed completeness, and that the last space must also contain $-\infty$, because all domains have the least element.

We will call these domains the *domains of lower estimates* for reasons, which will become apparent in the next section. We denote these domains as R^+ or $R^+_{[A,\, B]}$.

We can also consider a vertical segment of a real line $[A, B]$ with the inverse order, $\sqsubseteq = \geq$. We can also consider vertical rays $[A, +\infty]$, $[-\infty, A]$ and the whole line $[-\infty, +\infty]$ with the same order.

We will call these continuous lattices the *domains of upper estimates.* We denote these domains with the inverse order as R^- or $R^-_{[A,\,B]}$.

Note that when we talk about $R^+_{[A,\,B]}$ or $R^-_{[A,\,B]}$, the cases of $A = -\infty$ and/or $B = +\infty$ are included.

4.1.1 The "Way Below" Relation

One can easily see that $\bot_{R^+} = A$ and $\bot_{R^-} = B$.

It is also easy to see that given $a, b \in [A,\,B]$, $a \ll_{R^+} b$ iff $a = \bot_{R^+}$ or $a < b$. Similarly, $a \ll_{R^-} b$ iff $a = \bot_{R^-}$ or $a > b$.

This allows one to understand, why these domains are continuous but not algebraic.

Any subset K, dense in $[A,\,B]$ in the traditional sense, can be taken as a basis.

4.1.2 Scott Topology

Consider the domain R^+. Its Scott open sets are the empty set, the whole space R^+, and semi-intervals or open rays $(a,\,B], a > A$.

For the domain R^-, the Scott open sets are the empty set, the whole space R^-, and semi-intervals or open rays $[A,\,b), b < B$.

4.2 Interval Numbers

In this section we define the domain of interval numbers belonging to $[A,\,B]$, where the cases of $A = -\infty$ and/or $B = +\infty$ are included. This domain will consist of segments $[a,\,b]$, where $A \leq a \leq b \leq B$. We denote this domain as R^I or $R^I_{[A,\,B]}$.

Informally speaking, the segment $[a, b]$ is interpreted as a partially defined number x, about which it is known that

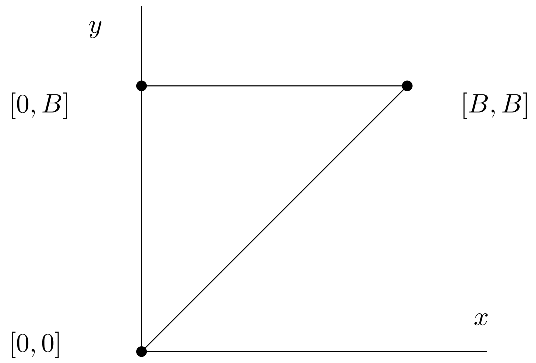
$$a \leq x \leq b.$$

If $x = [a, b]$, we call a a *lower bound* or *lower estimate* of x, and b an *upper bound* or *upper estimate* of x.

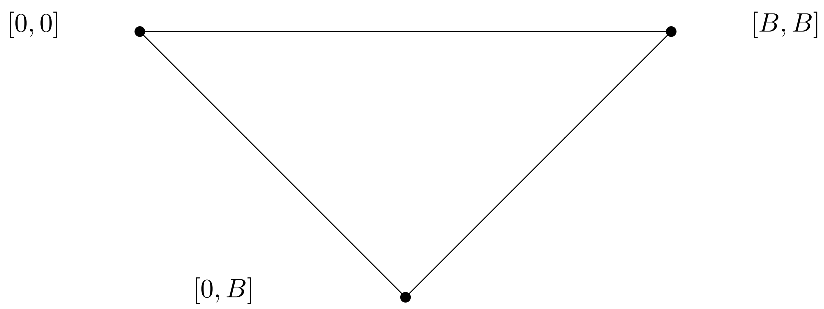
We can consider a product domain $R^+ \times R^-$ with coordinate-wise order. Then R^I can be defined as a subset of $R^+ \times R^-$, obtained by by elimination of pairs $\langle a, b\rangle$, such that $a > b$, from $R^+ \times R^-$. R^I inherits its partial order from $R^+ \times R^-$.

More specifically, if interval $[c, d]$ is within interval $[a, b]$, that is $a \le c \le d \le b$, than $[c, d]$ gives more information about a number, than $[a, b]$. In such a case, we say that $[a, b]$ *approximates* $[c, d]$, and write $[a, b] \sqsubseteq_{R^I} [c, d]$.

Consider the case, where $A = 0$ and B is finite. Interval numbers $[x, y]$ can be thought of as a triangle on the plane.



We flip this triangle so that the best defined interval numbers, like $[a, a]$, are on the top, and the least defined one, $[0, B]$, is on the bottom.



4.2.1 The "Way Below" Relation

One can easily see that $\perp^{R^I} = [A,\, B]$.

It also easy to see that $[a,\, b] \ll [c,\, d]$ iff either $a = A$ and $b = B$, or $a = c = A$ and $b > d$, or $b = d = B$ and $a < c$, or $a < c$ and $b > d$.

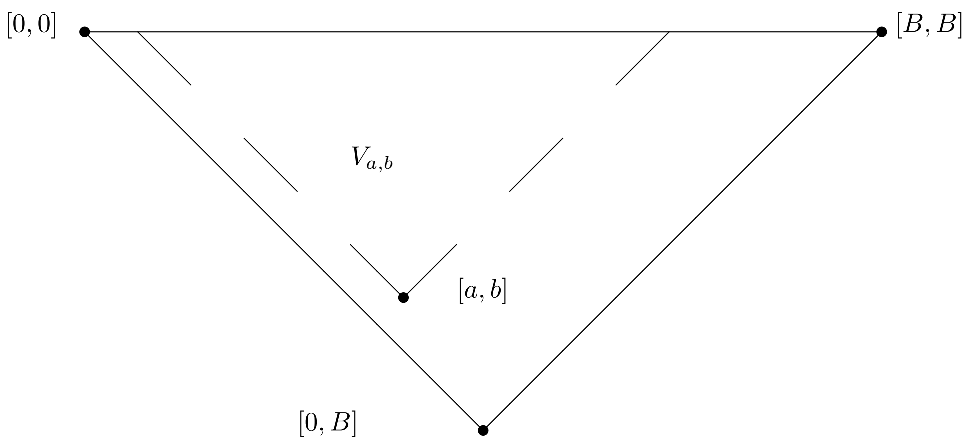
It is easy to see that R^I is a continuous Scott domain. Consider any set Q, dense in $[A,\, B]$ in the traditional sense, and take elements $[A,\, b]$, $[a,\, B]$, and $[a,\, b]$, such that $a < b$ and $a, b \in Q$, as a basis $K \subseteq R^I$.

4.2.2 Scott topology

We consider the basis K built above and build the base of Scott topology.

For segments $[A,\, b] \in K$ take sets $V_b^A = \{[x, y] \mid A \le x \le y < b\}$. For segments $[a,\, B] \in K$ take sets $V_a^B = \{[x, y] \mid a < x \le y \le B\}$. For segments $[a,\, b] \in K$ take sets $V_{a,b} = \{[x, y] \mid a < x \le y < b\}$.

Sets V_b^A, V_a^B, and $V_{a,b}$ form the base of Scott topology.



Chapter 5

Algebraic Information Systems and Domains

This chapter presents the approach of information systems in the case of algebraic Scott domains as can be found, e.g. in [48, 35, 36].

The algebraic information systems and domains described in this chapter reflect the standard notions of inference and theories in traditional formal theories.

5.1 Information Systems and Domains

The approach of information systems describes an approximation domain as a set of *theories* in a logical calculus. There are two ways to follow this approach. One could consider a given approximation domain, which obeys certain specific axioms, and then try to build a logical calculus such that its theories form an isomorphic domain. This approach is very useful, when one needs to investigate which class of domains is covered by a specific variant of information systems, or when one starts with a given domain to begin with.

However, for didactic purposes, a different approach has proven much more valuable. This approach was used in the first key paper on information systems by Dana

Scott [48] and works as follows. We presume that there is some approximation domain on the background, but we do not specify it precisely. Then, having this hypothetical domain in mind, we reason about the desired properties of our logical systems and impose appropriate axioms describing the behavior of these systems. Then we define domains as sets of theories and establish their properties as theorems rather than postulating them as axioms.

5.1.1 UNKNOWN as a Truth Value

An information system is a *logical calculus* of *elementary statements* and their *finite conjuctions*. These elementary statements and finite conjunctions should be viewed as *continuous predicates* of a special kind on the elements of the domain in question. These predicates map domains elements to a two-element set of truth values, however these truth values are not ordinary **true** and **false**, but **true** and **unknown**. This crucial feature is usually not emphasized enough, but one needs to keep it in mind.

The reason for this distinction is that we view a domain element, x, as a dynamic object, about which only some of the information is known at any given time of the computational process, but which later can be supplemented with more information, and thus replaced with y, $x \sqsubseteq y$. The information, which was **unknown** about x, can become **true** about y. This dynamic process is, however, viewed as *monotonic* with respect to time — once a piece of information becomes **true** about an element, it remains this way further on.

It is possible to talk about **false** pieces of information about x — these are such pieces of information which are not **true** about any y, such that $x \sqsubseteq y$, that is, the pieces which cannot become **true** about x during its arbitrary monotonic evolution. The **false** truth values will be treated via *consistency* mechanism, however they will always remain auxiliary, and usually can be easily eliminated from the scene if necessary, while **true** and **unknown** truth values are essential.

5.1.2 Example: Formal Theories

The most natural example one should keep in mind is any traditional formal theory, where elementary statements are all formulas, and the domain in question is the domain of all theories ordered by ordinary set-theoretic inclusion.

If a statement belongs to a theory, we will say that it is **true** in (or, if you wish, "about") this theory, otherwise it is **unknown** in this theory. If the theory cannot be refined to a larger non-contradictory theory to include a specific statement, we might wish to say that this statement is **false** in ("about") this theory.

In the traditional formal theories any deduction is thought of as a formal text of finite length. Hence any entailment of a statement or a contradiction from a set of statements is made on the basis of a finite subset of this set. This *finitarity* property will be reflected in the definitions below.

5.1.3 Consistency and Entailment

Let us denote the set of *elementary statements* as D, and the set of all its finite subsets as $\mathcal{P}_{\text{fin}}(D)$. We interpret a finite subset, $u \subseteq D$, $u = \{d_1, \cdots, d_n\}$, as a *finite conjuction* of statements $d_1, \cdots, d_n$.

Keeping the hypothetical approximation domain at the background, we would like to call a finite conjuction, u, of elementary statements *consistent*, if there is a domain element, x, about which all statements of u are true.

It is traditional to assume that $\emptyset$ is consistent, that is, that the domain in question is non-empty [48]. We do follow this tradition here. It is also traditional to assume that any single elementary statement is true about some domain element, that is, $\{d\}$ is consistent for any elementary statement d. This means that one considers contradictory elementary statements to be "junk" and wants to exclude them from an information system. We would like to depart from this convention, as it seems to gain nothing, and makes it more difficult to talk about some natural examples, like the one considered in the previous subsection, and also makes it impossible to talk about effective structures on domains in full generality (the point of view, advocated, in particular, by

V.Yu.Sazonov; our style of information system is, in fact, intermediate between the traditional one and the style of Sazonov [43] and seems to be the most convenient).

We would like to say, that a finite conjuction, u, of elementary statements *entails* an elementary statement, d, if whenever u is true about a domain element, x, statement d is also true about x.

Again, it is traditional to consider only consistent conjuctions, u, in the context of entailment. However, it is much more convenient to assume that a conjuction, which is not consistent, entails everything. It is technically convenient to introduce a special **false** statement, ∇, which entails everything and follows from all inconsistent conjunctions.

Taking all this into account, the following definition becomes quite natural.

Definition 5.1.1 The tuple $A = (D_A, \nabla_A, \vdash_A)$, where D_A is a set of distinctive *tokens* (*elementary statements*), $\nabla_A \in D_A$ (the *false statement*), $\vdash_A \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_A)$ (the *entailment* relation) is called an *algebraic information system* if

1. $\emptyset \not\vdash_A \{\nabla_A\}$ (non-degeneracy; a calculus must admit at least one non-contradictory theory);

2. $\forall u, v \in \mathcal{P}_{\text{fin}}(D_A).\ v \subseteq u \Rightarrow u \vdash_A v$ (reflexivity of entailment and conjunction elimination);

3. $\forall u, v_1, \ldots, v_n, w \in \mathcal{P}_{\text{fin}}(D_A).\ u \vdash_A v_1, \ldots, u \vdash_A v_n, v_1 \cup \ldots \cup v_n \vdash_A w \Rightarrow u \vdash_A w$ (transitivity of entailment and conjunction introduction);

4. $\forall u \in \mathcal{P}_{\text{fin}}(D_A).\ \{\nabla_A\} \vdash_A u$ (contradiction entails everything).

The equivalence between this and the classical definition of information system will be shown in Subsection 5.1.7.

5.1.4 Theories as Domain Elements

Definition 5.1.2 *Domain* $|A|$ associated with an algebraic information system A is the set of theories,

$\{x \subseteq D_A \mid$

1. $u \subseteq x, u \vdash_A v \Rightarrow v \subseteq x$ (x is deductively closed);

2. $\nabla_A \notin x$ (deductively closed x is consistent) }.

Informally, we will say that if $d \in x$, then d is **true** about x, otherwise d is **unknown** about x.

5.1.5 Properties of Domains

Assume that algebraic information system A is given. Domain $|A|$ is partially ordered by set-theoretical inclusion, $\sqsubseteq_A = \subseteq_{D_A}$. We study the properties of this order.

5.1.5.1 Deductive Closure and the Least Element

Definition 5.1.3 Subset $x \subseteq D_A$ is called (finitely) *consistent*, if for any finite $u \subseteq x$, $u \not\vdash_A \{\nabla_A\}$.

Definition 5.1.4 Given subset $x \subseteq D_A$, we call subset $\overline{x} \subseteq D_A$, $\overline{x} = \{d \in D_A \mid$ there is finite $u \subseteq x$, such that $u \vdash_A \{d\}\}$, the *deductive closure* of x.

Lemma 5.1.1 *If subset $x \subseteq D_A$ is consistent, then $\overline{x}$ is a domain element.*

Proof. This simple proof is still quite instructive, so we go through it here. First of all, we want to show that $\overline{x}$ is deductively closed, that is if $u \subseteq \overline{x}$ and $u \vdash_A v$ then $v \subseteq \overline{x}$.

Formula $u \vdash_A v$ implies finiteness of u and v, hence consider $u = \{d_1, \ldots, d_n\}$. By definition of $\overline{x}$, there are $u_1, \ldots, u_n \subseteq x$, such that $u_1 \vdash_A \{d_1\}, \ldots, u_n \vdash_A \{d_n\}$. Consider $u' = u_1 \cup \ldots \cup u_n$. Then, combining the rule of conjuction elimination and the rule of transitivity, we obtain $u' \vdash_A \{d_1\}, \ldots, u' \vdash_A \{d_n\}$, applying the same combination of rules in their conjuction introduction and reflexivity incarnations once again, we obtain $u' \vdash_A u$, applying transitivity once again we obtain $u' \vdash_A v$, and from this and $u' \subseteq x$ one concludes that $v \subseteq \overline{x}$.

Consistency follows straightforwardly from the two previous definitions. $\Box$

Lemma 5.1.2 *$x \subseteq y \Rightarrow \overline{x} \subseteq \overline{y}$.*

Lemma 5.1.3 *Since $\emptyset$ is consistent, $\perp_A = \overline{\emptyset}$ is the least domain element.*

5.1.5.2 Directed Completeness

Lemma 5.1.4 *Domain $|A|$ associated with an algebraic information system A is directed complete.*

Proof. Consider a directed $S \subseteq |A|$. For the case of $S = \emptyset$, Lemma 5.1.3 establishes the the existence of $\perp_A = \sqcup\emptyset$.

Assume that $S = \{s_i \mid i \in I\}$ is a non-empty directed set of elements of domain $|A|$. Let us establish that $\bigcup_{i\in I} s_i$ is an element of $|A|$. From this it would be easy to see that $\sqcup_{i\in I} s_i = \bigcup_{i\in I} s_i$.

First, establish the deductive closeness of $\bigcup_{i\in I} s_i$. Assume that $u = \{d_1, \ldots, d_n\} \subseteq \bigcup_{i\in I} s_i$. Then $d_1 \in s_{i_1}, \ldots, d_n \in s_{i_n}$. By directness of $(s_i, i \in I)$, there is such $i \in I$, that $s_{i_1} \subseteq s_i, \ldots, s_{i_n} \subseteq s_i$. Then $u \subseteq s_i$, and by deductive closeness of s_i, if $u \vdash_A v$, then $v \subseteq s_i$. Hence if $u \vdash_A v$, then $v \subseteq \bigcup_{i\in I} s_i$.

Consistency is trivial. □

5.1.5.3 "Way Below" Relation and Algebraicity

Lemma 5.1.5 *Given an algebraic information system, A, an element, x, of associated domain $|A|$ is* compact *if and only if there is a finite set, $u \subseteq D_A$, such that $x = \overline{u}$.*

Proof. Assume, that $x = \overline{\{d_1, \ldots, d_n\}}$ and that $x \sqsubseteq \sqcup S$, where S is a directed set. By the proof of Lemma 5.1.4, $\sqcup S = \bigcup_{s\in S} s$, hence, since $d_1, \ldots, d_n \in x$, there are $s_1, ..., s_n \in S$, such that $d_1 \in s_1, \ldots, d_n \in s_n$. Hence, by directness of S, there is $s \in S$, such that $d_1 \in s, \ldots, d_n \in s$. Hence, by the deductive closeness of s, $x \subseteq s$, and $x \sqsubseteq s$.

Conversely, assume that x cannot be represented as a deductive closure of its finite subset. Then consider set $U = \{u \mid u \subseteq x, u \text{ finite}\}$. It is easy to see, that $S = \{\overline{u} \mid u \in U\}$ is a directed set, and that $x = \sqcup S$, but for no $u \in U$, $x \sqsubseteq \overline{u}$, hence x is not compact. □

The second part of the proof of the previous Lemma implies that the set of compact elements characterized there possesses the properties of a basis of dcpo and, hence, the following Lemma holds.

Lemma 5.1.6 *Domain $|A|$ associated with algebraic information system A is an algebraic dcpo.*

5.1.5.4 Bounded Completeness

Lemma 5.1.7 *Domain $|A|$ associated with algebraic information system A is bounded complete.*

Proof. Consider $Y \subseteq |A|$, such that there is $x \in |A|$, such that for all $y \in Y$, $y \sqsubseteq x$. Then $\bigcup_{y \in Y} y$ is a consistent set, and $\overline{\bigcup_{y \in Y} y}$ is the desired least upper bound of Y. $\square$

5.1.5.5 Algebraic Scott Domains

The discource of this section can be summarized by the following Theorem.

Theorem 5.1.1 *The domain $|A|$ associated with algebraic information system A is an algebraic Scott domain.*

5.1.6 Representing Algebraic Scott Domains

Here we show that algebraic information systems describe exactly algebraic Scott domains.

Theorem 5.1.2 *For any algebraic Scott domain $(X, \sqsubseteq_X)$, there is an algebraic information system A, such that the partial order $(|A|, \sqsubseteq_A)$ is isomorphic to partial order $(X, \sqsubseteq_X)$.*

Proof. Denote the set of compact elements of X as K. Take $D_A = K \cup \{\nabla_A\}$, where ∇_A is a new token. For finite subsets $u, v \subseteq D_A$, we say that $u \vdash_A v$ if

1. either u is unbounded subset of X, or $\nabla_A \in u$;

2. or u has an upper bound in X, and for any $k \in v$, $k \sqsubseteq_X \sqcup_X u$.

First, we must prove that A is an algebraic information system. It is a simple proof, we only show transitivity axiom here, which is only non-trivial if u has an upper bound in X. Then assumption, that $u \vdash_A v_1, \ldots, u \vdash_A v_n$, means that any element k of any of these v_i's is less or equal to $\sqcup_X u$. In turn, $v_1 \cup \ldots \cup v_n \vdash_A w$, implies that all elements of w are under the least upper bound of all those k's. But $\sqcup_X u$ is (some) upper bound of those k's, hence all elements of w are under $\sqcup_X u$.

Now we have to establish an order-preserving isomorphism between X and $|A|$. With every $x \in X$ we associate $K_x = \{k \in K \mid k \sqsubseteq x\}$. Now we are going to prove that all K_x are elements of $|A|$, and every element of $|A|$ equals to K_x for some $x \in X$. Then, due to the fact that $x \sqsubseteq y \Leftrightarrow K_x \subseteq K_y$, the relation $x \leftrightarrow K_x$ would yield the necessary isomorphism.

It is easy to establish that K_x is an element, by using the directness of K_x.

In the opposite direction, consider $S \subseteq K$, which is an element of $|A|$. First of all, let us show that S is directed. By consistency of S any finite $u \subseteq S$ is bounded. By bounded completeness of X, there is $\sqcup_X u$. Then, from the directness of $K_{\sqcup_X u}$, we immediately obtain the existence of compact $k \in K_{\sqcup_X u}$, which is an upper bound of u (of course, this immediately implies $k = \sqcup_X u$, i.e. the least upper bound of a finite bounded set of compact elements is compact). By definition of $\vdash_A$, $u \vdash_A \{k\}$. Hence, the deductive closeness of S implies that S is directed.

Now, using the fact that X is directed complete, consider $x \in X$, such that $x = \sqcup_X S$. Let us prove that $S = K_x$. The only non-trivial direction is to prove $K_x \subseteq S$. Consider compact element $k \in K_x$. The definition of "way-below" relation and of element x, together with the facts $k \ll k$ and $k \sqsubseteq x$, imply that there is $k' \in S$, such that $k \sqsubseteq k'$. Then deductive closeness of S implies $k \in S$. $\square$

5.1.7 Equivalence between Classical Information Systems and Our Definition

Definition 5.1.5 $A = (D'_A, Con'_A, \vdash'_A)$, where D'_A is a set of distinctive *tokens* (*assertions*), $Con'_A \subseteq \mathcal{P}_{\text{fin}}(D'_A)$ (the *consistent* finite conjunctions of assertions), $\vdash'_A \subseteq Con'_A \times D'_A$ (the *entailment* relation), is called a *classical information system* if

(i) $\emptyset \in Con'_A$ (non-degeneracy; empty domains are barred);

(ii) $\forall d \in D'_A . \{d\} \in Con'_A$ (no "junk");

(iii) $\forall u \subseteq v \in Con'_A . u \in Con'_A$;

(iv) $\forall d \in u \in Con'_A . u \vdash'_A d$ (reflexivity of entailment);

(v) $\forall u \in Con'_A, d \in D'_A . u \vdash'_A d \Rightarrow u \cup \{d\} \in Con'_A$ (entailment preserves consistency);

(vi) $\forall u, v \in Con'_A, d \in D'_A . (\forall d' \in v . u \vdash'_A d'), v \vdash'_A d \Rightarrow u \vdash'_A d$ (transitivity of entailment).

The domain $|A|$ associated with a classical information system A is defined as a set of $x \in D'_A$ satisfying the consistency condition, $\forall u \subseteq x.\ u$ is finite $\Rightarrow u \in Con'_A$, and the condition of deductive closeness, $\forall u \subseteq x, d \in D'_A.\ u \vdash'_A d \Rightarrow d \in x$. Of course, $\sqsubseteq_{|A|}$ is defined to equal $\subseteq$.

All such domains are algebraic Scott domains, and for any algebraic Scott domain X it is possible to build a classical information system A, such that $|A|$ is isomorphic to X. So everything is very similar to our version of algebraic information systems.

Now we are going to establish an even stronger equivalence between these two versions. We are going to build two translations as follows. Given an algebraic information system $A = (D_A, \nabla_A, \vdash_A)$, we will build a classical information system, $B = C(A)$, and given a classical information system, $B = (D'_B, Con'_B, \vdash'_B)$, we will build an algebraic information system $A = S(B)$, such that the following properties hold.

For domains, $|C(A)| = |A|$ and $|S(B)| = |B|$, where the set equalities take place, as opposed to mere isomorphisms. For information systems, $C(S(B)) = B$ and if we consider $A_c = S(C(A))$, then $D_{A_c} = D_A \setminus \{d \in D_A \mid d \vdash_A \nabla_A \& d \neq \nabla_A\}$ and $\vdash_{A_c}$ is obtained by restricting $\vdash_A$ on $\mathcal{P}_{\text{fin}}(D_{A_c}) \times \mathcal{P}_{\text{fin}}(D_{A_c})$. Basically, the only disturbance that $S \circ C$ cannot avoid is that other "junk" tokens equivalent to ∇_A die.

The translations are defined by the following formulas.

$D'_{C(A)} = \{d \in D_A \mid \{d\} \not\vdash_A \nabla_A\}$; $Con'_{C(A)} = \{u \in \mathcal{P}_{\text{fin}}(D_A) \mid u \not\vdash_A \nabla_A\}$; if $u \in Con'_{C(A)}, d \in D'_{C(A)}$, then $u \vdash'_{C(A)} d \Leftrightarrow u \vdash_A \{d\}$.

$D_{S(B)} = D'_B \cup \{\nabla_{S(B)}\}$ (under assumption $\nabla_{S(B)} \notin D'_B$). Consider $u, v \in \mathcal{P}_{\text{fin}}(D_{S(B)}), d_1, \ldots, d_n \in D_{S(B)}$. If $u \notin Con'_A$ then $u \vdash_{S(B)} v$. If $u \in Con'_A$ then $u \vdash_{S(B)} \{d_1, \ldots, d_n\} \Leftrightarrow u \vdash'_B d_1, \ldots, u \vdash'_B d_n$.

It is easy to check that if A is an algebraic information system and B is a classical information system, then $C(A)$ is a classical information system, $S(B)$ is an algebraic information system, and our claims above hold.

5.2 Approximable Mappings and Continuous Functions

Information systems allow to think about domain elements as theories. Likewise, this approach allows to think about Scott continuous functions as special inference engines, which infer information about output, $f(x)$, from information about input, x.

Unfortunately, these *input-output inference engines* are called *approximable mappings* for their property to approximate one another. While this is an important property, Scott continuous functions thought of as graphs, or, in fact, any functions to domains approximate one another. Moreover, the "mappings" in question are not even functions. However, at this point of the development of the field one does not have much of a choice, but to follow the accepted terminology.

The following definition takes into account the intuition, that consistent information about input should produce consistent information about output, and that "native" inference relations of input and output domains can be used by an input-output inference engine.

Definition 5.2.1 Given two information systems, A and B, relation $f \subseteq \mathcal{P}_{\text{fin}}(D_A) \times$

$\mathcal{P}_{\text{fin}}(D_B)$ is called an *input-output inference relation* or an *approximable mapping* between A and B, if the following axioms hold:

1. $\emptyset f \emptyset$ (non-triviliality; minimalistic version of $uf\emptyset$);

2. $\forall u \in \mathcal{P}_{\text{fin}}(D_A).\ u \not\vdash_A \{\nabla_A\} \Rightarrow \neg(uf\{\nabla_B\})$ (preservation of consistency);

3. $\{\nabla_A\}f\{\nabla_B\}$ (inconsistency about an input allows to infer anything about the corresponding output);

4. $\forall u \in \mathcal{P}_{\text{fin}}(D_A), v_1, \ldots, v_n \in \mathcal{P}_{\text{fin}}(D_B).\ ufv_1, \ldots ufv_n \Rightarrow uf(v_1 \cup \ldots \cup v_n)$ (accumulation of output information, i.e. conjunction introduction);

5. $\forall u', u \in \mathcal{P}_{\text{fin}}(D_A), v, v' \in \mathcal{P}_{\text{fin}}(D_B).\ u' \vdash_A u, ufv, v \vdash_B v' \Rightarrow u'fv'$ (transitivity with "native" inference relations in A and B).

5.2.1 Scott Continuous Functions

An approximable mapping, f, between information systems A and B naturally gives rise to a function, $|f| : |A| \to |B|$, where $|f|(x)$ is computed by inferring all information from x using f.

We will see that $|f|$ is always Scott continuous, and that for any Scott continuous function $g : |A| \to |B|$ one can find an approximable mapping $\widehat{g}$ between A and B, such that $|\widehat{g}| = g$, $\widehat{|f|} = f$. Together with the results about identity maps and composition this yields an equivalence between the category of algebraic information systems and approximable mappings and the category of algebraic Scott domains and Scott continuous functions.

Definition 5.2.2 Given an approximable mapping f between A and B, define the associated function $|f| : |A| \to |B|$ by formula $|f|(x) = \{d \in D_B \mid \exists u \subseteq x.\ uf\{d\}\}$.

Theorem 5.2.1 *The function $|f|$ is correctly defined and Scott continuous.*

Proof. First, of all, to prove correctness we need to show, that $|f|(x) \in |B|$. We need to show deductive closeness and consistency of $|f|(x)$. Assume, that $v =$

$\{d_1, \ldots, d_n\} \subseteq |f|(x)$. Then there are $u_1, \ldots, u_n \subseteq x$, such that $u_1 f \{d_1\}, \ldots, u_n f \{d_n\}$. Then, by axioms of information systems and approximable mappings, ufv, where $u = u_1 \cup \ldots \cup u_n$. Hence, if $v \vdash_B v'$, by axioms of approximable mapping ufv', from which it is easy to infer that $v' \subseteq |f|(x)$. Likewise, if ∇_B would belong to $|f|(x)$, there would be some $u \subseteq x$, $uf\{\nabla_B\}$, contradicting the combination of consistency of x and, hence, u, and the preservation of consistency axiom for f.

Monotonicity of $|f|$ is obvious. Given a directed set $S \subseteq |A|$, we show that $|f|(\sqcup_A S) = \bigcup_{s \in S} |f|(s)$, thus establishing preservation of least upper bounds of directed sets and Scott continuity. The less trivial part is to show $|f|(\sqcup_A S) \subseteq \bigcup_{s \in S} |f|(s)$. Recall that $\sqcup_A S = \bigcup_{s \in S} s$. Assume, that $d \in |f|(\bigcup_{s \in S} s)$, that is there is $v = \{d_1, \ldots, d_n\} \in \bigcup_{s \in S} s$, such that $vf\{d\}$. Then there are $s_1, \ldots, s_n \in S$, such that $d_1 \in s_1, \ldots, d_n \in s_n$. Then, by directness of S, we can find such $s \in S$, that $v \subseteq s$. Hence $d \in |f|(s)$ and belongs to the union in question. $\Box$

Definition 5.2.3 Given a Scott continuous function $g : |A| \to |B|$, define the associated relation $\widehat{g} \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_B)$ by formula $u\widehat{g}v \Leftrightarrow \forall x \in |A|.\ u \subseteq x \Rightarrow v \subseteq g(x)$.

It is easy to check that $\widehat{g}$ is an approximable mapping.

If $u \in \mathcal{P}_{\text{fin}}(D_A)$ is consistent, then $u\widehat{g}v \Leftrightarrow v \subseteq g(\overline{u})$, otherwise $u\widehat{g}v$ for all $v \in \mathcal{P}_{\text{fin}}(D_B)$.

This and the Scott continuity of g allow to establish easily, that $|\widehat{g}| = g$, $\widehat{|f|} = f$.

Moreover, $f \subseteq g$ iff $|f| \sqsubseteq_{[A \to B]} |g|$.

5.2.2 Identity and Composition

It is easy to see that $|\vdash_A| = id_A$, where $\forall x \in |A|.\ id_A(x) = x$.

Define the *composition* of approximable mappings $f \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_B)$ and $g \subseteq \mathcal{P}_{\text{fin}}(D_B) \times \mathcal{P}_{\text{fin}}(D_C)$ as $g \circ f = h \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_C)$, such that $uhw \Leftrightarrow \exists v \in \mathcal{P}_{\text{fin}}(D_B).\ ufv, vgw$.

One can think of $g \circ f$ as an input-output inference engine obtained by hooking the input of engine g to the output of engine f: $\left(\quad \leftarrow g \leftarrow f \leftarrow \quad \right)$.

It is easy to check that $|g \circ f| = |g| \circ |f|$.

5.3 Functional Spaces

Consider algebraic information systems A and B. We are going to define an information system, $[A \to B]$, such that $|[A \to B]|$ would consist precisely of all approximable mappings between A and B.

Since there is an isomorphism between approximable mappings and Scott continuous functions, and since the set-theoretical inclusion of approximable mappings corresponds to the partial order on the domain of Scott continuous functions, $[|A| \to |B|]$, we will use the information system $[A \to B]$ to represent this functional space.

Take $D_{[A\to B]} = \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_B)$. Take $\nabla_{[A\to B]} = \langle \emptyset, \{\nabla_B\}\rangle$.

Say, that $\{\langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle\} \vdash_{[A\to B]} \{\langle u'_1, v'_1\rangle, \ldots, \langle u'_m, v'_m\rangle\}$, where $n, m \geq 0$, if

1. for any $i \in \{1, \ldots, m\}$, $\bigcup_{\{j\in\{1,\ldots,n\} \mid u'_i \vdash_A u_j\}} v_j \vdash_B v'_i$ or $u'_i \vdash_A \nabla_A$; or

2. $\exists I \subseteq \{1, \ldots, n\}$. $\bigcup_{i\in I} u_i \not\vdash_A \nabla_A, \bigcup_{i\in I} v_i \vdash_B \nabla_B$.

We leave the necessary correctness checks to the reader.

5.4 Effective Domains and Computations

We say that an algebraic information system A is *effective*, if D_A and $\vdash_A$ are recursively enumerable. The corresponding domain $|A|$ is called *effective* too.

An important example is produced by the *effective domain of arithmetic theories* in any of the usual systems of arithmetic. This example shows why our degree of generality is the right one.

Usually people give a more restrictive definition, which is equivalent to D_A and $\vdash_A$ being recursive. One of the reasons for this is the inconvenience of the definition of a classical information system. Indeed, if A is an effective algebraic information system, then $Con'_{C(A)}$ is co-recursively enumerable, and $\vdash'_{C(A)}$ has to be defined effectively on a co-recursively enumerable domain of definition, which is not a trivial undertaking, and, in any case, the result would be awkward.

An element $x \in |A|$ is called *computable* if it is a recursively enumerable set. Note that this condition is equivalent to the recursive enumerability of K_x under our construction from Section 5.1.6.

If f is a computable element of a functional domain, $|[A \to B]|$, we call it a *computable function*. Observe that computable functions map computable elements to computable elements and, moreover, transform a recursive enumeration of x into the recursive enumeration of $|f|(x)$.

We should note here that the problems of the correctness of definition for effective domains and computable elements are not decidable in general. For example, it is often impossible to develop a procedure deciding whether a given effective domain is not empty, or whether a given recursive enumeration of a computable element does not contain ∇.

An implementation of a computable element is some computational device, which recursively enumerates the tokens of this element. The issues of more effective implementation of some classes of computable elements (defined by some restricted classes of formulas) are very important from the practical viewpoint and constitute Problem A.

Part II

Logic of Fixed Points for Domains

Chapter 6

Non-reflexive Logics for Non-algebraic Domains

In the previous chapter we saw that information systems based on the ordinary logic correspond to algebraic Scott domains. Hence, in order to generalize the logical approach to larger classes of dcpo's, one has to modify the logic of inference and/or the notion of theory.

In this chapter we concern ourselves with bounded complete dcpo's. Some progress for some classes of non-bounded complete spaces was achieved by Abramsky [1].

This chapter develops our ideas from [8, 9] and Appendix 1 of [10]

6.1 Non-reflexive Logic

The seminal paper [28] by R.Hoofman generalized the logical approach to non-algebraic continuous domains. Hoofman replaced the ordinary relexivity rule, $A \vdash A$, with a weaker property of *inverse modus ponens*: $A \vdash C \Rightarrow \exists B.\ A \vdash B, B \vdash C$. His paper is a well-written one, but at the same time the actual and simple reasons for his construction to work are hidden in its later sections. Thus, the average reader gets an impression that it is a miracle construction, and the real intuition behind this paper is lost.

In particular, another important feature, which allows to exploit non-relexivity, is underemphasized. This feature is the change in the notion of a theory (i.e. domain element). A theory, x, is traditionally a consistent, deductively closed set of statements. Hoofman adds an additional requirement of *inverse deductive closeness*, which is trivial for a reflexive situation: $\forall d \in D_A.d \in x \Rightarrow \exists u \subseteq_{fin} x.u \vdash_A d$.

We will soon see, that this feature is essential for his approach, while the *inverse modus ponens rule* is important only in order to maintain continuity of the resulting domain. In fact, we generalize the results of [28] to arbitrary *spaces of fixed points of Scott continuous transformations of algebraic Scott domains*, by omitting both the inverse and the standard rule of *modus ponens*.

6.1.1 Correspondence between Properties of Scott Continuous Functions and Inference Rules

Scott continuous functions are equivalent to input-output inference engines (approximable mappings). In particular, given an algebraic information system A, Scott continuous transformations $|f|$ of the corresponding domain $|A|$ are equivalent to the generalized inferences $f \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_A)$, which are transitive with the standard $\vdash_A$ and respect consistency and inconsistency.

6.1.1.1 Reflexivity

Consider the reflexive f, $v \subseteq u \Rightarrow ufv$. Since $ufv, v \vdash_A w \Rightarrow ufw$, if f is reflexive, then from $\forall u \in \mathcal{P}_{\text{fin}}(D_A).\ ufu$ we can infer $\vdash_A \subseteq f$. Hence if f is reflexive, then $\forall x \in |A|.\ x \sqsubseteq_A |f|(x)$.

Conversely, $\forall x \in |A|.\ x \sqsubseteq_A |f|(x)$ implies that f is reflexive.

6.1.1.2 Transitivity

It is easy to see, that $\forall u, v, w \in \mathcal{P}_{\text{fin}}(D_A).\ ufv, vfw \Rightarrow ufw$ is equivalent to the condition $|f| \circ |f| \sqsubseteq_{[A \to A]} |f|$.

6.1.1.3 Inverse Transitivity

It is easy to see, that the Hoofman rule of inverse transitivity, $\forall u, w \in \mathcal{P}_{\text{fin}}(D_A).\ ufw \Rightarrow \exists v \in \mathcal{P}_{\text{fin}}(D_A).\ ufv, vfw$ is equivalent to $|f| \sqsubseteq_{[A \to A]} |f| \circ |f|$.

6.1.1.4 Retractions

The previous two paragraphs imply that f is transitive and inversely transtive if and only if $|f|$ is a retraction: $|f| = |f| \circ |f|$.

6.1.1.5 Inverse Reflexivity

The rule $\forall x \in |A|.\ |f|(x) \sqsubseteq_A x$ is equivalent to the following rule of "inverse reflexivity": $ufv \Rightarrow u \vdash v$, which will be transformed into $ufv \Rightarrow v \subseteq u$ in some simple cases.

6.1.1.6 Closures and Projections

It is easy to see, that reflexivity implies inverse transitivity, and that inverse reflexivity implies transitivity.

Hence, the combination of reflexivity and transitivity yields precisely *closures* (such retractions $|f|$, that $x \sqsubseteq_A |f|(x)$), and the combination of inverse reflexivity and inverse transitivity yields precisely *projections* (such retractions $|f|$, that $|f|(x) \sqsubseteq_A x$).

6.1.2 Fixed Points as Theories

Consider the definition of $|f|$: $|f|(x) = \{d \mid \exists u \in x.\ uf\{d\}\}$. So $|f|(x)$ consists of tokens which are inferrable from x via inference engine f.

Hence, the deductive closeness of x with respect to inference f, $\forall u, v \in \mathcal{P}_{\text{fin}}(D_A)$. $u \in x, ufv \Rightarrow v \in x$ is equivalent to $|f|(x) \sqsubseteq_A x$.

Similarly, the *inverse deductive closeness* of x with respect to inference f, namely $\forall v \in \mathcal{P}_{\text{fin}}(D_A).\ v \in x \Rightarrow \exists u \in \mathcal{P}_{\text{fin}}(D_A).\ u \in x, ufv$, is equivalent to $x \sqsubseteq |f|(x)$.

Hence, together the deductive closeness and the inverse deductive closeness of x with respect to inference f is equivalent to x being a fixed point of $|f|$: $|f|(x) = x$.

6.1.3 Domains of Fixed Points

The previous paragraph suggests the following procedure. Consider an algebraic information system A and replace its entailment relation with an approximable mapping f between A and A.

Then replace the notion of theory with the consistent subset $x \subseteq D_A$, such that x is deductively closed and inversely deductively closed with respect to f. The result is the domain $|A_f|$ of fixed points of contionuous transformation $|f|$, $|A_f| = \mathrm{Fix}(f) \subseteq |A|$.

We can call $|A_f|$ a *fixed-point subdomain* of $|A|$.

However, we want to introduce a more general notion of a fixed-point subdomain.

6.1.4 Adding the Top Element

Consider an algebraic information system $A = (D_A, \nabla_A, \vdash_A)$ and the corresponding domain $|A|$. Consider an element $\nabla_{A_\top} \notin D_A$ and define the algebraic information system $A_\top$ as follows.

Take $D_{A_\top} = D_A \bigcup \{\nabla_{A_\top}\}$. Of course, we take $\nabla_{A_\top}$ as the ∇ of the new system. We set $u \vdash_{A_\top} v$ iff either $u \vdash_A v$ or $\nabla_{A_\top} \in u$.

Then $|A_\top| = |A| \bigcup \{\top\}$, where $\top = D_{A_\top} = \overline{\{\nabla_A\}}$ is the new top element.

6.1.5 General Notion of Fixed-Point Subdomain

In this chapter we consider approximable mappings as generalized entailment relations. Sometimes we would like an approximable mapping f to entail the contradiction from some of finite conjuctions of statements from D_A, noncontradictory under $\vdash_A$. In order to formalize such a situation we have to consider approximable mappings f from A to $A_\top$.

The general notion of a fixed-point subdomain of A is the set of fixed points of Scott continuous function $|f| : |A| \to |A_\top|$, $\mathrm{Fix}(f) = \{x \in A \mid x = |f|(x)\}$. Equivalently, one can consider fixed points of Scott continuous transformations $|f|$ of domain $A_\top$, such that $|f|(\top) = \top$.

If, for the sake of tradition, one wants to impose the requirement that the subdomains are non-empty, one has to add the requirement $|f|(\bot) \neq \top$.

6.1.6 Application: Removing the Compact Top Element

Assume that an algebraic Scott domain $|B|$ has the compact top element $\top_B \neq \bot_B$ (in particular, $|A_\top|$ has the compact top element, $\overline{\{\nabla_{A_\top}\}}$). Then consider $|f| : |B| \to |B_\top|$, such that $|f|(\top_B) = \top_{B_\top}$ and $|f|(x) = x$ for all other $x \in |B|$.

Then $\mathrm{Fix}(f) = B \setminus \{\top_B\}$.

The compactness of $\top_B$ is important, since a Scott continuous function on an algebraic Scott domain is completly defined by its values on compact elements.

In terms of entailment, ufv iff either $u \vdash_B v$ or $\top_B \subseteq \overline{u}$. It is easy to see, that since $\top_B \neq \bot_B$, the resulting system $B_{\not\top} = (D_B, \nabla_B, f \bigcap(\mathcal{P}_{\mathrm{fin}}(D_B) \times \mathcal{P}_{\mathrm{fin}}(D_B)))$ is an algebraic information system, and $|B_{\not\top}| = B \setminus \{\top_B\}$.

6.1.7 Powersets and Qualitative Domains

Consider set D, such that $\nabla_A \notin D$, and define $D_A = D \bigcup \{\nabla_A\}$ and $u \vdash_A^m v$ iff $v \subseteq u$ or $\nabla_A \in u$. The resulting *minimal* algebraic information system $A^m = (D_A, \nabla_A, \vdash_A^m)$ defines the powerset of D as its domain $|A^m|$.

If we consider $W \subset \mathcal{P}_{\mathrm{fin}}(D_A)$, such that $\emptyset \notin W$, $\{\nabla_A\} \in W$, and modify the previous construction so that $u \vdash_A v$ iff $v \subseteq u$ or $\exists w \in W.\ w \subseteq u$, then we obtain a *qualitative domain* (see [28]). We can modify the construction of cutting the compact top element above to obtain any qualitative domain from the powerset domain.

6.1.8 General Notion of Finitary Information System And Finitary Scott Domain

Consider an algebraic information system $A = (D_A, \nabla_A, \vdash_A)$ and an approximable mapping f from A to $A_\top$. We agreed that f describes a fixed-point subdomain of $|A|$. Observe that f also describes a fixed-point subdomain of $|A^m|$. Moreover, the sets of fixed points of corresponding Scott continuous functions $|f| : |A| \to |A_\top|$ and $|f| : |A^m| \to |A^m_\top|$ coincide.

This leads us to a general definition of what we call a *finitary information system* and the definition of the corresponding *finitary Scott domain*. We will respect the rule $|f|(\bot) \neq \top$ meaning the non-emptyness of the resulting domains. These definitions will be respectively a streamlined equivalent of A^m together with an approximable mapping f between A^m and $A^m_\top$ and a description of the set of fixed points of the corresponding function $|f|$.

Definition 6.1.1 The tuple $A = (D_A, \nabla_A, \vdash'_A)$, where D_A is a set of distinctive *tokens* (*elementary statements*), $\nabla_A \in D_A$ (the *false statement*), $\vdash'_A \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_A)$ (the *entailment* relation) is called a *finitary information system* if

1. $\emptyset \not\vdash'_A \{\nabla_A\}$ (non-degeneracy; a calculus must admit at least one non-contradictory theory);
2. $\emptyset \vdash'_A \emptyset$ (non-triviliality);
3. $\forall u, v_1, \ldots, v_n \in \mathcal{P}_{\text{fin}}(D_A).\ u \vdash'_A v_1, \ldots u \vdash'_A v_n \Rightarrow u \vdash'_A (v_1 \cup \ldots \cup v_n)$ (accumulation of output information);
4. $\forall u', u, v, v' \in \mathcal{P}_{\text{fin}}(D_A).\ u' \subseteq u, u' \vdash'_A v', v \subseteq v' \Rightarrow u \vdash'_A v$;
5. $\forall u, v \in \mathcal{P}_{\text{fin}}(D_A).\ u \vdash'_A \{\nabla_A\} \Rightarrow u \vdash'_A v$;
6. $\forall u, v \in \mathcal{P}_{\text{fin}}(D_A).\ \nabla_A \in u \Rightarrow u \vdash'_A v$.

Definition 6.1.2 The *finitary Scott domain* $|A|$ associated with a finitary information system A is the set of theories,

$\{x \subseteq D_A \mid$

1. $u \subseteq x, u \vdash'_A v \Rightarrow v \subseteq x$ (x is deductively closed);
2. $v \subseteq x \Rightarrow \exists u \in \mathcal{P}_{\text{fin}}(D_A).\ u \vdash'_A v, u \subseteq x$ (x is inversely deductively closed);
3. $\nabla_A \notin x$ (deductively closed x is consistent) $\}$.

In order to check the correctness of our discourse, one has to consider information systems $A^m = (D_A, \nabla_A, \vdash^m_A)$ and the corresponding system $A^m_\top$. Given a finitary information system $A = (D_A, \nabla_A, \vdash'_A)$, one should define $f \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_A \bigcup \{\nabla_{A^m_\top}\})$ via $ufv \Leftrightarrow u \vdash'_A v$ or $\nabla_A \in u$ and establish that f is an approximable mapping, such that $\neg(\emptyset f \{\nabla_{A^m_\top}\})$. Then given an approximable mapping f between A^m and $A^m_\top$, such that $\neg(\emptyset f \{\nabla_{A^m_\top}\})$, one should define $\vdash'_A$ via $u \vdash'_A v \Leftrightarrow ufv$ and $\nabla_{A^m_\top} \notin v$ and establish that $A = (D_A, \nabla_A, \vdash'_A)$ is a finitary information system.

Finally one should observe that we have just defined one-to-one correspondence between all possible entailment relations in finitary information systems $(D_A, \nabla_A, \vdash'_A)$ and all approximable mappings between A^m and $A^m_\top$, such that $\neg(\emptyset f \{\nabla_{A^m_\top}\})$, and that under that correspondence finitary Scott domains are exactly the sets of fixed points of the respective functions $|f|$.

6.1.9 Domains of Fixed-Point Subdomains

In the subsections 6.1.5 and 6.1.8 we defined a finitary information system and the corresponding finitary Scott domain of fixed points of $|f|$ for any approximable mapping f, such that $\neg(\emptyset f \{\nabla_{A_\top}\})$, between any algebraic information system $A = (D_A, \nabla_A, \vdash_A)$ and the corresponding system $A_\top$.

The approximable mapping f, such that $\emptyset f \{\nabla_{A_\top}\}$, corresponding to Scott continuous function $|f|$, such that $|f|(\bot) = \top_{A_\top}$, is the top compact element of the domain $[A \to A_\top]$, and, hence, it can be removed by the technique described above, yielding the algebraic Scott domain $|[A \to A_\top]_{\overline{\top}}|$.

Because these mappings f are in one-to-one correspondence with fixed-point subdomains of $|A|$, we can take the domain $|[A \to A_\top]_{\overline{\top}}|$ as representing fixed-point subdomains of $|A|$. Of course, different f might have the same set of fixed points, and hence the corresponding fixed-point domains will coincide as sets, but their underlying entailment relations will differ, so they should be considered different subdomains.

Hence we call $|[A \to A_\top]_{\overline{\top}}|$ the *domain of fixed-point subdomains* of $|A|$.

6.1.10 Discussion

One should observe that f is not an arbitrary new entailment relation, but is closely related to the original $\vdash_A$, namely f is transitive with respect to $\vdash_A$ due to the axioms of approximable mappings. This property is responsible for the fact that all elements of the resulting fixed-point subdomain belong to the original domain.

We can think about a fixed-point subdomain as the result of some elements of the original domain being destroyed. There are three mechanisms of such a destruction. An element can lose its deductive closeness or consistency. When these two mechanisms are involved, we still remain within the realm of reflexive logic and algebraic subdomains result. This important case is covered in details in Chapter 7.

The third mechanism of destruction of elements of the original domain is the loss of inverse deductive closeness. When this happens we, in general, go beyond ordinary reflexive logic. However, there are cases of non-reflexive f's, where we still can remain within the realm of reflexive logic. The price for this is the distortion of the resulting subdomain: instead of the proper fixed-point subdomain we only obtain a domain isomorphic to this fixed-point subdomain. One such case, namely the case of finitary retractions, is considered in details in Chapter 8.

We could have considered a different definition of a fixed-point subdomain of $|A|$, namely any finitary information system determined by arbitrary approximable mapping f from A^m to $A^m_\top$, such that $\mathrm{Fix}(|f|)$ is a non-empty subset of $|A|$. Then, however, it is unlikely that we could form a domain of such subdomains, although I did not try to prove such a result. Even if we could do so, such domains of subdomains would not have to be isomorphic for isomorphic domains $|A|$ and $|B|$ if the cardinalities of D_A and D_B differ. Hence this alternative definition is unsatisfactory.

6.1.11 Results for Continuous Scott Domains

Now the results on continuous information systems by Hoofman can be easily explained. He retained transitivity of f and replaced its reflexivity by inverse transitivity. This resulted in $|f|$ being a retraction.

It is well known that retractions of continuous lattices are continuous lattices, and that all continuous lattices can be obtained as retractions of powersets, and these results can be generalized for continuous Scott domains, since the only thing which distinguish them from continuous lattices is that continuous Scott domains do not have to possess the top element.

These facts explain the success of Hoofman's program for continuous Scott domains.

6.2 Infinitary Logic

In [43] Sazonov and Sviridenko introduced another generalization of logic. They kept reflexivity and transitivity, but removed the finitarity requirement that an infinite set of statements x only infers those statements, which the finite subsets $u \subseteq x$ infer.

Their approach describes exactly all bounded complete partial orders. Directed completeness is equivalent to the following requirement of *partial finitarity*: a set of statements x cannot infer the contradiction, unless one of its finite subsets $u \subseteq x$ infers the contradiction.

Hence partially finitary systems of Sazonov and Sviridenko exactly correspond to complete Scott domains.

Sazonov and Sviridenko also gave characterizations of some subclasses of complete Scott domains in their system, in particular, they described a class of their logic corresponding to continuous Scott domains.

6.3 Open Issues

6.3.1 The Class of Finitary Scott Domains

What is the class of domains described as fixed-point domains is an important open question. It is well known that the set of fixed points of a Scott continuous transformation of a complete lattice is a complete lattice. It was traditionally thought that the converse is also true, namely that any complete lattice can be obtained as a set of fixed points of a Scott continuous transformation of a sufficiently large powerset. For example, Exercise 18.4.3(ii)(1) on page 491 of the famous textbook on lambda calculus by Barendregt [5] asks to establish this for the case of countable bases of the respective Scott topologies.

Hence we expected that our approach would allow us to obtain all complete Scott domains. We thus expected to follow the terminology of continuous information systems by Hoofman and call the fixed-point information systems developed in this chapter *complete information systems*.

However, our analysis of literature and numerous conversations with experts in the field convinced us that the problem stated in the textbook by Barendregt is, in fact, open. Our attempts to solve it were, so far, unsuccessful. Hence we opted for the less committing terminology of *finitary information systems* and *finitary Scott domains*.

6.3.2 Important Subclasses

What subclasses of finitary domains will be obtained, if we require f to be transitive or, more strongly, inversely reflexive? The conjecture is that we still obtain all finitary Scott domains as sets of fixed points of the corresponding functions $|f|$.

Transitivity is, of course, a very desirable property of a logical system, so it is of interest to check whether it restricts generality. These questions come in at least two flavors: for f defining a continuous transformation of $|A^m|$ and for f defining a continuous transformation of an arbitrary algebraic Scott domain $|A|$.

6.3.3 Approximable Mappings And Other Issues for Finitary Information Systems

Hoofman successfully introduced the notion of continuous approximable mapping for continuous information systems based on the fact, that given spaces A and B and their retractions $r_A : A \to A$ and $r_B : B \to B$, one can build a retraction $(A \to B) \to (A \to B)$, namely $f \mapsto r_B \circ f \circ r_A$. Scott continuous functions $f : A \to B$, such that $f = r_B \circ f \circ r_A$, are in one-to-one correspondence with Scott continuous functions

$\mathrm{Fix}(r_A) \to \mathrm{Fix}(r_B)$.

Unfortunately, no such simple solution seems possible for finitary information systems. Yet, a satisfactory notion of finitary approximable mapping seems to be necessary for further successful studies of finitary information systems.

There is a possibly simpler variant of this problem if f is required to be transitive.

6.3.4 Possible Translation between Nonreflexive and Nonfinitary Logics

In [43] Sazonov and Sviridenko gave a translation between their logic and the logic of Hoofman for continuous Scott domains.

Naturally there is a question whether such a translation is possible for a larger class of domains.

It also might be of interest to combine the features of nonreflexive and nonfinitary logic in one system.

Chapter 7

Subdomains for the Algebraic Case

This chapter represents our results obtained in 1986-1988 and presented in [7]. With the introduction of our new version of the definition of algebraic information system and our ideas in nonreflexive logic the presentation is greatly simplified.

For the duration of this chapter "domain" means algebraic Scott domain, "subdomain" means algebraic Scott subdomain, and "information system" means algebraic information system.

7.1 The Brief History of the Question

In this chapter, we study the question "Which subsets of a domain should be considered subdomains?" for the domains described by algebraic information systems. One criterion for a good answer is a requirement that all subdomains of domain $|A|$ should themselves form a domain, $|Sub(A)|$. We would also like to be able to solve domain equations such as $|X| \cong |Sub(X)| + \ldots$. It will be shown in the next few pages that the framework of domains as abstract cpo's does not provide sufficient hints for the solution, but information systems do.

The systematic use of the operation of taking a certain subset of a domain to form another domain and the systematic consideration of domains, whose elements are domains, first appear in the mid-'70's [47, 49]. It is notable that nobody has tried to consider arbitrary subsets of domain $|A|$ that would satisfy a particular version of domain axioms, although at the first glance this would seem to be the most intuitive and general version of a subdomain definition in the framework of domains as abstract cpo's. The reason why nobody has considered such a definition is that such arbitrary subsets might satisfy the domain axioms for random causes, and their collection would be totally unmanageable; in particular, one cannot hope to form a domain, $|Sub(A)|$, with these subsets as elements.

The next idea, which originated by Dana Scott in [47], is to define a subdomain as a set of fixed points of a *retraction* belonging to a certain class. There are two problems here: a) What are the reasons for using retractions? b) What is the appropriate class of retractions? None of these problems receives a clear answer in the framework of domains as abstract cpo's. On the use of retractions Scott writes: "it seems almost an accident that the idea [to use the retractions] can be applied" (see [47], p.540).

For the algebraic case, Scott suggests using *closure operations*, that is, retractions $|r|$, such that $|r| \sqsupseteq id$. This suggestion can be motivated by the theorem (see [47], Theorem 5.1) saying that the fixed points of a closure operation [on an algebraic lattice] form an algebraic lattice. Although this is not true for arbitrary retractions or projections (i.e. retractions $|r|$, such that $|r| \sqsubseteq id$), if we consider *finitary* retractions or projections (*finitary* means algebraicity of the set of fixed points), then this argument no longer favors closure operations.

In the paper entitled "Data Types as Objects" [49], Shamir and Wadge suggest that to describe the systems of polymorphic types, one should consider data types that incorporate their own subtypes as elements, and allow an element of a data type to belong simultaneously to different subtypes of this data type.

Let us briefly present the main features of subtypes in [49]. The subtype $|Y| = \{x \in |D| \mid x \sqsubseteq y\}$ is associated with each element y of a data type $|D|$. We can view such subtype as the set of fixed points of the (finitary, in the algebraic case) projection $x \mapsto x \sqcap y$. Subtypes are ordered simply by inclusion. Thus, this simple approach yields the result that $|D| \cong |Sub(D)|$ for every domain $|D|$. Each element represents the subtype of its approximations, and an element x belongs to all subtypes $|Y|$ of $|D|$ that are represented by y's, such that $x \sqsubseteq y$.

At the end of the Introduction to [7], we discussed the rather adverse relations between retraction-based approaches to subtyping ([47, 49], and this chapter are based on retractions) and the dominant form of modern theories of typing that deals with types of intensional objects (the type polymorphism in ML is, probably, the most widely known example).

7.2 Algebraic Subdomains Correspond to Closures

When we considered this problem in [7], we were using classical information systems. We wrote: "Let us consider informally, what should we do to an information system A to obtain a subdomain $|B| \subseteq |A|$. We would like to destroy some elements of $|A|$. The elements are consistent and deductively closed sets of assertions in A. Therefore, some elements of $|A|$ must lose their consistency or deductive closure to be destroyed. Also notice, that the loss of deductive closure means intensification of entailment."

Then we figured out, that inconsistency can be treated as entailment of contradiction in $A_\top$, and hence the loss of consistency can be also treated as strengthening of entailment. The need to keep considering separately entailment and consistency in a number of technical situations, nevertheless, complicated our whole discourse.

With the definition of algebraic information system introduced in this Thesis, the matters become much simpler.

Definition 7.2.1 We say that for algebraic information systems A and B, domain $|B|$ is an *(algebraic) subdomain* of domain $|A|$ if $|B| \subseteq |A|$.

Lemma 7.2.1 (Contravariance) *For algebraic information systems A and B, such that $D_A = D_B$, $|B| \subseteq |A|$ iff $\vdash_A \subseteq \vdash_B$.*

It is easy to see that the case when $D_A \neq D_B$ does not lead to new domains $|B|$, so we do not consider this case further.

Since every algebraic information system is a finitary information system, the discourse from the previous chapter applies. Namely, we can see that if $\vdash_A \subseteq \vdash_B$, the approximable mapping f from A to $A_\top$, which corresponds to $\vdash_B$, is reflexive.

Since B is an algebraic information system, the restriction of f to $A \to A$ approximable mapping (or its extension to $A_\top \to A_\top$) is also transitive, so the corresponding restriction or extension of $|f|$ would be a closure. Also the non-triviality condition $|f|(\perp_A) \neq \top_{A_\top}$ holds.

Definition 7.2.2 An approximable mapping f from an algebraic information system A to the algebraic information system $A_\top$ is called a *generalized non-trivial closure*, if

1. $\forall u, v \in_{fin} \mathcal{P}_{\text{fin}}(D_A).\ u \vdash_A v \Rightarrow ufv$ (reflexivity of the restriction on A);

2. $\forall u, v, w \in_{fin} \mathcal{P}_{\text{fin}}(D_A).\ ufv, vfw \Rightarrow ufw$ (transitivity of the restriction on A);

3. $\neg(\emptyset f \{\nabla_{A_\top}\})$ (non-triviality);

Lemma 7.2.2 *Such generalized non-trivial closures from $|[A \to A_\top]|$ are in one-to-one correspodence with algebraic subdomains of $|A|$. Moreover, for closures f and g and the corresponding subdomains $|F|$ and $|G|$, $|F| \subseteq |G|$ iff $g \subseteq f$.*

7.3 A Metatheorem on Reflexive Transitive Closure

We are about to build the domain of algebraic subdomains of an algebraic Scott domain $|A|$ as an algebraic subdomain of the algebraic Scott domain $|[A \to A_\top]|$. Our new setting allows us to do it in a systematic way compared to [7].

Definition 7.3.1 Consider the minimal algebraic information system $A^m = (D_A, \nabla_A, \vdash_A^m)$ and any binary relation $R \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_A)$. Then define the *reflexive transitive closure* $\vdash_A^R \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_A)$ as follows. For any $u, v \in \mathcal{P}_{\text{fin}}(D_A)$, we say that

$u \vdash_A^R v$, if there is a finite sequence $v_1, \ldots, v_n \in \mathcal{P}_{\text{fin}}(D_A)$, such that $v = v_n$ and the following condition holds:

Denote $u_0 = u$ and, inductively, $u_i = u_{i-1} \bigcup v_i$ for all $i \in \{1, \ldots, n\}$. Then for any $i \in \{1, \ldots, n\}$, we require that either $\nabla_A \in u_{i-1}$, or $v_i \subseteq u_{i-1}$, or there is $u' \subseteq u_{i-1}$, such that $u' R v_i$.

Definition 7.3.2 Consider the minimal algebraic information system $A^m = (D_A, \nabla_A, \vdash_A^m)$ and any binary relation $R \subseteq \mathcal{P}_{\text{fin}}(D_A) \times \mathcal{P}_{\text{fin}}(D_A)$. We say that a set $x \subseteq D_A$ is a *theory with respect to R*, if x is consistent, that is $\nabla_A \notin x$, and closed under R, that is if $u \subseteq x$ and uRv, then $v \subseteq x$.

Theorem 7.3.1 *Consider the minimal algebraic information system $A^m = (D_A, \nabla_A, \vdash_A^m)$ and any binary relation $R \subseteq \mathcal{P}_{fin}(D_A) \times \mathcal{P}_{fin}(D_A)$. If there is at least one closed theory x with respect to R, then $A^R = (D_A, \nabla_A, \vdash_A^R)$ is an algebraic information system and domain $|A^R|$ is equal to the set of all theories closed with respect to R.*

Corollary 7.3.1 *Consider an algebraic information system $A = (D_A, \nabla_A, \vdash_A)$ and any binary relation $R \subseteq \mathcal{P}_{fin}(D_A) \times \mathcal{P}_{fin}(D_A)$. Define binary relation $R' = R \bigcup \vdash_A$. Then if domain $|A|$ contains at least one element which is a theory closed with respect to R, then domain $|A^{R'}|$ consists of those elements of the domains $|A|$, which are theories closed with respect to R.*

7.4 The Domain of Algebraic Subdomains

We are now ready to study the domain of algebraic subdomains of an algebraic Scott domain $|A|$. We denote this domain $|Sub(A)|$. This domain will consist of all generalized non-trivial closures from $|[A \to A_\top]|$. These closures will represent the corresponding algebraic subdomains of $|A|$.

Theorem 7.4.1 *The set $|Sub(A)|$ of generalized non-trivial closures from $|[A \to A_\top]|$ is an algebraic subdomain of the algebraic Scott domain $|[A \to A_\top]|$. Hence, $|Sub(A)|$ is an algebraic Scott domain itself.*

Proof. Consider the algebraic information system $F = [A \to A_\top]$. We need to define the relation R, which would axiomatize properties of generalized non-trivial closures.

Consider the following three sets:

$R_r = \{\langle \emptyset, \{\langle u, v \rangle\} \rangle \mid u \vdash_A v, u, v \in \mathcal{P}_{\text{fin}}(D_A)\}$.

$R_t = \{\langle \{\langle u, v \rangle, \langle v, w \rangle\}, \{\langle u, w \rangle\} \rangle \mid u, v, w \in \mathcal{P}_{\text{fin}}(D_A)\}$.

$R_n = \{\{\emptyset, \nabla_{A_\top}\}, \{\nabla_F\}\}$.

They axiomatize, respectively, reflexivity, transitivity, and non-triviality.

Take $R = R_r \bigcup R_t \bigcup R_n$.

The set $\vdash_A$ is an approximable mapping from $|F| = |[A \to A_\top]|$. (Remark: $| \vdash_A |$ is the embedding $\lambda x.x$). $\vdash_A$ is also a theory with respect to R. Hence, the corollary from the previous section implies, that domain $|F^{R'}|$ consists of those elements of the domains $|F|$, which are theories closed with respect to R. It is easy to see that these elements are precisely all generalized non-trivial closures.

Hence, the information system $F^{R'}$ is exactly the information system $Sub(A)$ we have been looking for. $\Box$

Theorem 7.4.2 *For any $b, c \in |Sub(A)|$, $b \sqsubseteq_{Sub(A)} c$ iff for the corresponding algebraic subdomains $|B|$ and $|C|$, $|C| \subseteq |B|$.*

The idea behind this result is simple: the stronger the entailment, i.e. the larger the generalized closure is, the fewer elements of $|A|$ remain intact (consistent and deductively closed).

This differs favorably from [49] and from the other approaches to subdomains as sets of fixed points of projections, where subdomains are ordered by inclusion. Indeed, the smaller a subdomain is as a set, the more information we have about each of its elements, and hence, the larger this subdomain should be informationally ($integers \sqsupseteq reals$ is desirable).

Theorem 7.4.3 *The continuous functions $|i| : |A| \to |Sub(A)|$ and $|j| : |Sub(A)| \to |A|$ defined below yield a projection of $|Sub(A)|$ onto $|A|$. In terms of closure operations*

$$|i|(y) = \lambda x : |A|.x \sqcup_{A_\top} y;$$

$$|j|(r) = |r|(\bot).$$

In terms of subdomains themselves

$$|i|(y) = \{x \in |A| \mid x \sqsupseteq y\};$$

$$|j|(S) = \min_{|A|} S, \textit{ where } S \textit{ is a subdomain of } |A|.$$

Notice that some light non-triviality is involved in the construction of the embedding $|i| : |A| \to |Sub(A)|$. All one-element subsets of $|A|$ form subdomains (the corresponding generalized closure is $|r_y| = \lambda x : |A_\top|.if\ x \sqsubseteq y\ then\ y\ else\ \top_{A_\top}$). But due to Theorem 7.4.2, they are all incomparable (in fact, they are total elements in $|Sub(A)|$). Therefore, the function $x \mapsto \{x\}$ is not monotonic. This implies that this function is not Scott continuous and cannot be used as an embedding.

7.5 Open Issues

7.5.1 Reflexive Domain of Algebraic Subdomains

Because $|A|$ and $|Sub(A)|$ form an embedding-projection pair, we can use the inverse limit construction to obtain solutions of certain domain equations involving Sub [35, 36, 48], e.g. $|X| \cong |Sub(X)|$, $|X| \cong |Sub(X)| + |A|$, etc. Because $Sub(A)$ describes theories of all possible ways to intensify our means of entailment in A, it might be of interest to study *explicitly* the iterations of this construction, $Sub(Sub(A))$, $Sub^3(A)$, etc., and their limit D, where $|D| \cong |Sub(D)|$.

7.5.2 Relationship between Subdomains and Subtypes

In [7] we expressed the opinion that compile-time and inheritance-oriented approaches to types, like subclasses in object-oriented programming and variants of strongly normalizing typed lambda-calculi, are incompatible with retraction-based subtyping in domain theory. This opinion was further confirmed by our reading of Section 10.4.4 on partial equivalence relations as types for systems with subtyping in the textbook by Mitchell [40].

Mitchell explains that in such systems entailment relation for the supertype should, in general, be stronger, than entailment for the subtype. This and next chapters show that this is impossible for retractions yielding algebraic domains, at least as long as we stay with the reflexive transitive finitary logic.

Thus the issue of developing practical programming language with types conforming to “data types as objects” paradigm [49], where types are run-time first-class objects, and any compile-time type checking and inference is considered to be an optimizing partial evaluation, remains an important open issue. As long as denotational semantics of such a language is expressible via algebraic domains, the notion of subtyping should probably correspond to the notion of subdomain developed in this chapter.

Chapter 8

Finitary Retractions and Projections

This chapter represents our results obtained in 1986-1988 and presented in [8]. With the introduction of our new version of the definition of algebraic information system and our ideas in nonreflexive logic this material became much more transparent.

In general, sets of fixed points of retractions of algebraic Scott domains form continuous Scott domains. In this chapter we study the classes of finitary retractions and projections.

Definition 8.0.1 A retraction or projection of an algebraic Scott domain $|r| : |A| \to |A|$ is called *finitary* if its set of fixed points is algebraic.

For the duration of this chapter "domain" means algebraic Scott domain, "subdomain" means algebraic Scott subdomain, and "information system" means algebraic information system.

8.1 Characterization of the Classes of Finitary Retractions and Projections

8.1.1 Finitary Retractions

Lemma 8.1.1 *A retraction $|r|$ of an algebraic Scott domain $|A|$ is finitary if and only if there is an algebraic Scott domain $|B|$ and Scott continuous functions $|i| : |B| \to |A|$ and $|j| : |A| \to |B|$, such that $\langle |i|, |j| \rangle$ is an embedding-retraction pair, such that $|r| = |i| \circ |j|$.*

Proof. It is enough to establish, that the embedding $\mathrm{Fix}(|r|) \to |A|$ and the map $R : |A| \to \mathrm{Fix}(|r|)$, where $R(x) = |r|(x)$ for all $x \in |A|$, are Scott continuous. This, in turn, follows from the fact that given a directed set S of fixed points of $|r|$, the least upper bound of S in $|A|$ is also a fixed point of $|r|$. $\square$

Definition 8.1.1 We say that a retraction $|r|$ of an algebraic Scott domain $|A|$ possesses the *intermediate reflexive property* if

$$\forall u, w \in \mathcal{P}_{\text{fin}}(D_A).\ urw \Rightarrow \exists v \in \mathcal{P}_{\text{fin}}(D_A).\ urv, vrv, vrw.$$

Lemma 8.1.2 *Consider a retraction $|r|$ of an algebraic Scott domain $|A|$ possessing the intermediate reflexive property. Define $A^r = (D^r_A, \nabla^r_A, \vdash^r_A)$ as follows. Let $D^r_A = \{u \in \mathcal{P}_{fin}(D_A) \mid uru\}$. Let $\nabla^r_A = \{\nabla_A\}$. Let $\{u_1, \ldots, u_n\} \vdash^r_A \{v_1, \ldots, v_m\}$ iff*

$(u_1 \bigcup \ldots \bigcup u_n) r (v_1 \bigcup \ldots \bigcup v_m)$ *for all* $u_1, \ldots, u_n, v_1, \ldots v_m \in D^r_A$.

Then A^r is an algebraic information system and $|A^r| \cong \mathrm{Fix}(|r|)$.

Proof. Checking the axioms of algebraic information system for A^r is straightforward. Observe that reflexivity of $\vdash^r_A$ follows from our selection of only such elements $u \in \mathcal{P}_{\text{fin}}(D_A)$ as members of D^r_A, that uru.

Now we use the intermediate reflexive property of r. For any $x \in \mathrm{Fix}(|r|)$, the set $\{u \in D^r_A \mid u \subseteq x\}$ belongs to $|A^r|$. For any $y \in |A^r|$, the set $\bigcup\{u \mid u \in y\}$ is a fixed point of $|r|$. These are monotonic injective maps. This establishes the desired isomorphism. $\square$

Lemma 8.1.3 *Given an algebraic information system A and an approximable mapping r from A to A defining a retraction $|r|$, the following conditions are equivalent:*

1. *$|r|$ is finitary.*

2. *$\forall u, w \in \mathcal{P}_{\mathit{fin}}(D_A).\ urw \Rightarrow \exists v \in \mathcal{P}_{\mathit{fin}}(D_A).\ urv, vrv, vrw.$*

3. *$\forall u, w \in \mathcal{P}_{\mathit{fin}}(D_A).\ urw \Rightarrow \exists v \in \mathcal{P}_{\mathit{fin}}(D_A).\ urv, vrv, v \vdash_A w.$*

Proof. $3 \Rightarrow 2$ follows from r being an approximable mapping, and $2 \Rightarrow 3$ follows from considering $v' = v \bigcup w$ and observing that urv, vrv, vrw implies $urv', v'rv', v' \vdash_A w$.

$2 \Rightarrow 1$ follows from the previous lemma.

Here we prove $1 \Rightarrow 2$. Consider an algebraic information system B and approximable mappings i and j from the Lemma 8.1.1. Consider $u, w \in \mathcal{P}_{\text{fin}}(D_A)$, such that urw. Then there is $v' \in \mathcal{P}_{\text{fin}}(D_B)$, such that ujv' and $v'iw$, because $r = i \circ j$. Because $j \circ i = \vdash_B$, there is $v \in \mathcal{P}_{\text{fin}}(D_A)$, such that $v'iv$ and vjv'. Then urv, vrv, and vrw. □

8.1.2 Finitary Projections

Lemma 8.1.4 *Given an algebraic information system A and an approximable mapping p from A to A, the following conditions are equivalent:*

1. *$|p|$ is a finitary projection.*

2. *$\forall u, w \in \mathcal{P}_{\mathit{fin}}(D_A).\ upw \Rightarrow \exists v \in \mathcal{P}_{\mathit{fin}}(D_A).\ u \vdash_A v, vpv, v \vdash_A w.$*

Proof. $1 \Rightarrow 2$ follows from the previous lemma and the fact that for a projection $upv \Rightarrow u \vdash_A v$.

To prove $2 \Rightarrow 1$, observe that in the condition 2 upv and vpw, so $|p|$ is a finitary retraction, and $u \vdash_A w$, so $|p|$ is a projection. □

8.2 Domains of Fixed Points of Finitary Projections and Retractions from the Viewpoint of Logic of Fixed Points

We could have used our technique of fixed-point subdomains to describe the set of fixed points of an arbitrary or finitary retraction or projection. However, since arbitrary retractions and projections do not possess reflexivity property, this would move us outside of the realm of algebraic information system even in the finitary case.

Instead, we use the fact that finitary retractions are exactly those possessing the *intermediate reflexive property* that if $\forall u, w \in \mathcal{P}_{\text{fin}}(D_A).\ urw \Rightarrow \exists v \in \mathcal{P}_{\text{fin}}(D_A).\ urv, vrv, vrw$. Informally, this means that there are sufficiently many finite conjuctions v, which are reflexive (vrv) with respect to the retraction r, considered as prospective entailment relation, so that any entailment done by r can be done *via* such a reflexive conjuction v. These reflexive conjunctions serve as a backbone of the new algebraic information system A^r describing the domain isomorphic to the domain of fixed points of $|r|$.

Then, in the case of r defining a projection, we build the domain $|A^r|$ via mechanism of *conjuctive completion* and of *forgetting* the statements, not reflexive with respect to r. In the case of r defining a general finitary retraction, we also have to replace the native entailment by the one induced by r (Lemma 8.1.2). It is precisely the *forgetting* step, which makes it impossible to consider sets of fixed points of finitary projections and retractions as algebraic subdomains, as we will see later in this section.

8.2.1 Conjuctive Completeness

We say that algebraic information system A is *(finitely) conjunctively complete*, if for any $u \in \mathcal{P}_{\text{fin}}(D_A)$, there is $d \in D_A$, such that $\{d\} \vdash_A u$ and $u \vdash \{d\}$.

The step $D_A^r = \{u \in \mathcal{P}_{\text{fin}}(D_A) \mid uru\}$ in Lemma 8.1.2 provides for *conjuctive completion*. Actually, $A_A^{\vdash_A}$ yield precisely the conjuctive completion of algebraic information system A, and $|A_A^{\vdash_A}| \cong |A|$.

It is actually enough to add conjuctions only for such $u \in \mathcal{P}_{\text{fin}}(D_A)$, that uru

and there is no such $d \in D_A$, that $\{d\} \vdash_A u$ and $u \vdash \{d\}$, so if we start from a conjuctively complete system to begin with, the conjuctive completion step can be omitted.

8.2.2 Domains of Fixed Points of Finitary Projections

When p defines a finitary projection, the construction of Lemma 8.1.2 is rewritten as follows.

An algebraic information system $A^p = (D_A^p, \nabla_A^p, \vdash_A^p)$ is defined by $D_A^p = \{u \in \mathcal{P}_{\text{fin}}(D_A) \mid upu\}$, $\nabla_A^p = \{\nabla_A\}$, $\{u_1, \ldots, u_n\} \vdash_A^p \{v_1, \ldots, v_m\}$ iff $(u_1 \bigcup \ldots \bigcup u_n) \vdash_A (v_1 \bigcup \ldots \bigcup v_m)$ for all $u_1, \ldots, u_n, v_1, \ldots v_m \in D_A^p$.

As before, $|A^p| \cong \text{Fix}(|p|)$.

If the information system A was conjuctively complete to begin with, the construction above could be modified as follows:

$D_A^p = \{d \in D_A \mid \{d\}p\{d\}\}$, $\nabla_A^p = \nabla_A$, $u \vdash_A^p v \Leftrightarrow u \vdash_A v$ for all $u, v \in \mathcal{P}_{\text{fin}}(D_A^p)$.

Then for any $x \in \text{Fix}(|p|)$, the set $x \bigcap D_A^p$ would belong to $|A^p|$. For any $y \in |A^p|$, the deductive closure of y in A is a fixed point of $|p|$. This establishes the desired isomorphism. In this case, for any $x \in |A|$, $|p|(x)$ is the deductive closure of $x \bigcap D_A^p$ in A.

8.2.3 Why Finitary Projections Do Not Form Algebraic Subdomains

We give two examples. The first example illustrates the need for conjunctive completeness or conjuctive completion. The second example actually shows, why forgetting does not allow us to consider sets of fixed points of finitary projections to be subdomains. In both examples we only list domain elements and sets of fixed points and leave it to the reader to restore the actual information systems and projections.

Consider the domain $|A| = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Consider the projection $|p|$ of $|A|$ onto $\{\emptyset, \{1, 2\}\}$. While there is an informaton system describing the domain $\{\emptyset, \{1, 2\}\}$, it cannot be obtained by a projection from our original domain, because the original domain is not conjunctively complete. The first version of our construction would allow us to obtain the domain isomorphic to $\mathrm{Fix}(|p|)$ as $|A^p| = \{\emptyset, \{\{1,2\}\}\}$ via a conjunctive completion.

On the other hand, if we consider an isomorphic conjunctively complete situation, $\{\emptyset, \{1\}, \{2\}, \{1,2,t\}\}$ and its projection onto $\{\emptyset, \{1,2,t\}\}$, the second version of our construction of A^p would yield $\{\emptyset, \{t\}\}$.

The set $\{\emptyset, \{1,2\}\}$ in the previous example can still be obtained as a subdomain of $|A|$, although not via a projection. Now we consider a different example, namely $|A| = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}\}$, and its projection $|p|$ onto $\{\emptyset, \{1,2\}, \{1,3\}\}$. The set $\{\emptyset, \{1,2\}, \{1,3\}\}$ is not a domain determined by any algebraic information system (consider, what would be $\overline{\{1\}}$). The set $\{\emptyset, \{1,2\}, \{1,3\}\}$ is not a subdomain of $|A|$ and cannot be obtained from $|A|$ via a closure operation (consider, where would $\{1\}$ go under a closure). This is a conjunctively closed situation, and the second version of our construction of A^p yields $|A^p| = \{\emptyset, \{2\}, \{3\}\}$.

The last example shows that sometimes a set of fixed points of a finitary projection is not equal to any domain described by an algebraic information system and, hence, cannot be considered an algebraic subdomain of the original domain. In other cases, like our first example, the set in question is an algebraic subdomain, but the closure operation describing it does not have anything in common with the finitary projection in question.

8.2.4 Domains of Fixed Points of Generalized Nontrivial Finitary Retractions

In order to fully incorporate the discourse of the previous two chapters into the framework of finitary retractions, one has to consider *generalized nontrivial retractions* $|r| : |A| \to |A_\top|$, such that $|r|(\bot) \neq \top_{A_\top}$ and $|r|(|r|(x)) = |r|(x)$, when $|r|(x) \neq \top_{A_\top}$.

In order to use the setup of Lemma 8.1.1, one has to consider retraction $|r'| : |A_\top| \to |A_\top|$, such that $|r'|(\top_{A_\top}) = \top_{A_\top}$ and $|r'|(x) = |r|(x)$ for $x \in |A|$. Notice that $|B|$ has the compact top element, and that $\mathrm{Fix}(|r|) \cong |B_{\not\top}|$.

Definition 8.1.1 can be extended to the generalized retraction $|r|$ without change.

Lemma 8.1.2 and its proof are applicable to the generalized nontrivial retraction $|r|$ without change as well.

Lemma 8.1.3 also holds for $|r|$, but in order to establish $1 \Rightarrow 2$ in its proof, one should consider $|r'|$.

Lemma 8.1.3 gives the criterion of finitarity of generalized nontrivial retractions, and Lemma 8.1.2 yields the construction of the domain isomorphic to the domain of fixed points of a generalized nontrivial finitary retraction.

8.2.5 Other Criteria of Finitarity

The following criterion of finitarity for projections was known before. A projection $|p| : |A| \to |A|$ is finitary iff $\forall x \in |A|.\ x = |p|(x) \Rightarrow x = \sqcup\{x_0 \in |A|_0 \mid x_0 = |p|(x_0), x_0 \sqsubseteq x\}$. This criterion can be obtained from Lemma 1(ii) of [29], which in effect says that the set of finite elements in $\mathrm{Fix}(|p|)$ is $\{x_0 \in |A|_0 \mid x_0 = |p|(x_0)\}$.

The consideration of information systems easily produces a much nicer criterion based on the intermediate reflexive property (Lemma 8.1.4). It does not require consideration of arbitrary non-finite elements x and allows to select finitary projections from the space of all continuous functions rather than from the space of projections. It can be literally rewritten in term of abstract cpo's: $\forall x_0, z_0 \in A_0.\ z_0 \sqsubseteq |p|(x_0) \Rightarrow \exists y_0 \in A_0.\ x_0 \sqsupseteq y_0 \sqsupseteq z_0, y_0 = |p|(y_0)$. Yet, one is unlikely to chose this criterion if he does not consider information systems because it involves inequality $z_0 \sqsubseteq |p|(x_0)$. Also notice that in this case we have to consider $|p|(x_0)$, which is generally not finite.

For retractions, the criterion based on the intermediate reflexive property is given by Lemma 8.1.3. In terms of abstract cpo's our criterion can be rewritten as follows: a retraction $|r| : |A| \to |A|$ is finitary iff $\forall x_0, z_0 \in A_0.\ z_0 \sqsubseteq |p|(x_0) \Rightarrow \exists y_0 \in A_0.\ y_0 \sqsubseteq |p|(x_0), y_0 \sqsubseteq |p|(y_0), z_0 \sqsubseteq |p|(y_0)$.

The finite elements of $\mathrm{Fix}(|r|)$ are obtained as images of the finite elements of $|A|$ under $|r|$. Hence, in the style close to [29], another criterion can be written as follows: a retraction $|r| : |A| \to |A|$ is finitary iff $\forall x \in |A|.\ x = |r|(x) \Rightarrow x = \sqcup\{|r|(x_0) \mid x_0 \in |A|_0, x_0 \sqsubseteq |r|(x_0) \sqsubseteq x\}$.

8.3 Domains of Finitary Projections and Generalized Finitary Retractions

8.3.1 Domain of Finitary Projections

The set of all finitary projections $|A| \to |A|$ can be obtained as the set of fixed points of the finitary projection $|Pr|$ of $|A \to A|$. Specifically, if $g \in |A \to A|$ define $f = |Pr|(g)$ by ufw iff $\exists v \in \mathcal{P}_{\mathrm{fin}}(D_A).\ u \vdash_A v, vgv, v \vdash_A w$.

It is easy to check that f is a finitary projection, and that if g is a finitary projection then $g = |Pr|(g)$. To prove that $|Pr|$ itself is a continuous function and a finitary projection, one should notice that $\{(u_1, w_1), \ldots, (u_n, w_n)\} Pr \{(u, w)\}$ iff $\exists v.\ u \vdash_A v, v \vdash_A w, \{(u_1, w_1), \ldots, (u_n, w_n)\} \vdash_{A \to A} (v, v)$.

8.3.2 Domain of Generalized Finitary Retractions

In a similar fashion the set of all generalized nontrivial finitary retractions can be obtained as a set of fixed points of a generalized nontrivial finitary retraction $|Ret|$ of $|A \to A_\top|$. Hence this set is an algebraic Scott domain.

8.4 Omitted Issues

The details of construction of $|Ret|$ is omitted in the present text. This construction is the result of mating the construction of $|Pr|$ with the construction of the domain of subdomains.

8.4.1 Limits of Projective Sequences

It is possible to generalize the construction of the limits of some projective sequences of domains defined by algebraic information systems, $\{\langle |A_n|, |A_{n+1}|, |i_n|, |j_n| \rangle \mid |i_n| : |A_n| \to |A_{n+1}|; |j_n| : |A_{n+1}| \to |A_n|; \langle |i_n|, |j_n| \rangle$ is an embedding-projection pair; $n = 1, \ldots\}$, from [35, 36] to arbitrary sequences of this kind. We do not give the details of the generalized construction here.

8.5 Open Issues

8.5.1 Issues Related to the Domain of Finitary Retractions

I do not know, whether $|Ret|$ is unique.

The question might be somewhat related to the properties of the space of all retractions, say, of a powerset. This space is a complete lattice, but it is not algebraic and not even continuous.

8.5.2 Other Subclasses of Finitary Retractions

Here we look at two special subclasses of the class of finitary retractions, which may deserve special attention.

Let us look at the criteria for finitarity of retractions and projections once more. For retractions it is $urw \Rightarrow \exists v.\ urv, vrv, vrw$, which can be rewritten as $urw \Rightarrow \exists v.\ urv, vrv, v \vdash w$. For projections it is $upw \Rightarrow \exists v.\ u \vdash v, vrv, v \vdash w$. There is an intermediate case — retractions satisfying the property that $urw \Rightarrow \exists v.\ u \vdash v, vrv, vrw$. This class of retractions includes both finitary projections and closure operations, and the retractions from this class possess a certain weak property of minimality, hence they can be called *quasi-minimal*.

Another class is obtained from the observation that while for finitary projections finite elements of sets of fixed points are finite in the original domain too, this does not generally hold for finitary retractions. We can call the retractions possessing this property *strongly finitary*. The class of strongly finitary retractions includes the class of finitary projections, but does not include the class of closure operations.

Both quasi-minimal retractions and strongly finitary retractions may be worth more detailed studies.

Part III

Elements of Analysis on Domains

Chapter 9

Co-continuous Valuations

In this chapter we present our joint results with Svetlana Shorina [15, 16]. We study the continuous normalized valuations μ on the systems of open sets of domains and introduce notions of *co-continuity*, $\{U_i,\, i \in I\}$ is a filtered system of sets $\Rightarrow \mu(\mathrm{Int}(\bigcap_{i\in I} U_i)) = \inf_{i\in I} \mu(U_i)$, and *strong non-degeneracy*, $U \subset V$ are open sets $\Rightarrow \mu(U) < \mu(V)$, for such valuations. We call the resulting class of valuations CC-valuations. The central result of this chapter is a construction of CC-valuations for Scott topologies on all continuous dcpo's with countable bases.

It seems that the notion of co-continuity of valuations and this result for the case of continuous Scott domains with countable bases are both new and belong to us [15, 16]. The generalization of this result to continuous dcpo's with countable bases belongs to Klaus Keimel [32]. He worked directly with *completely distributive lattices* of Scott open sets of continuous dcpo's and used the results about completely distributive lattices obtained by Raney in the fifties (see Exercise 2.30 on page 204 of [24]). Here we present a proof which can be considered a simplification of both our original proof and the proof obtained by Keimel. This proof also works for all continuous dcpo's with countable bases. A part of this proof, as predicted by Keimel, can be considered as a special case of Raney's results mentioned above. However, our construction is very simple and self-contained.

Keimel also pointed out in [32] that our results are quite surprising, because both *co-continuity* and *strong non-degeneracy* seem contradictory, as neither of them can hold for the system of open sets of the ordinary Hausdorff topology on $[0, 1]$. However, if we replace the system of open sets of this Hausdorff topology with the system of open intervals, both conditions would hold. We believe that the reason behind our results is that the Scott topology is coarse enough for its system of open sets to exhibit behaviors similar to the behaviors of typical *bases of open sets* of Hausdorff topologies.

9.1 CC-valuations

Consider a topological space $(X, \mathcal{O})$, where $\mathcal{O}$ consists of all open subsets of X. The following notions of the theory of valuations can be considered standard (for the most available presentation in a regular journal see [19]; the fundamental text in the theory of valuations on Scott opens sets is [31]).

Definition 1.1. A function $\mu : \mathcal{O} \to [0, +\infty]$ is called *valuation* if

1. $\forall U, V \in \mathcal{O}.\ U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;

2. $\forall U, V \in \mathcal{O}.\ \mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;

3. $\mu(\emptyset) = 0$.

Definition 1.2. A valuation μ is *bounded* if $\mu(X) < +\infty$. A valuation μ is *normalized* if $\mu(X) = 1$.

Remark: If a valuation μ is bounded and $\mu(X) \neq 0$, then it is always easy to replace it with a normalized valuation $\mu'(U) = \mu(U)/\mu(X)$.

Definition 1.3. Define a *directed system of open sets*, $\mathcal{U} = \{U_i,\ i \in I\}$, as satisfying the following condition: for any finite number of open sets $U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{U}$ there is U_i, $i \in I$, such that $U_{i_1} \subseteq U_i, \cdots, U_{i_n} \subseteq U_i$.

Definition 1.4. A valuation μ is called *continuous* when for any directed system of open sets $\mu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mu(U_i)$.

We introduce two new properties of valuations.

Definition 1.5. A valuation $\mu : \mathcal{O} \to [0, +\infty]$ is *strongly non-degenerate* if $\forall U, V \in \mathcal{O}.\, U \subset V \Rightarrow \mu(U) < \mu(V)$.

This is, obviously, a very strong requirement, and we will see later that it might be reasonable to look for weaker non-degeneracy conditions.

Consider a decreasing sequence of open sets $U_1 \supseteq U_2 \supseteq \ldots$, or, more generally, a *filtered system of open sets* $\mathcal{U} = \{U_i, i \in I\}$, meaning that for any finite system of open sets $U_{i_1}, \cdots U_{i_n} \in \mathcal{U}$ there is U_i, $i \in I$, such that $U_i \subseteq U_{i_1}, \cdots, U_i \subseteq U_{i_n}$. Consider the interior of the intersection of these sets. It is easy to see that for a valuation μ

$$\mu(\mathrm{Int}(\bigcap_{i \in I} U_i)) \le \inf_{i \in I} \mu(U_i).$$

Definition 1.6. A valuation μ is called *co-continuous* if for any filtered system of open sets $\{U_i,\, i \in I\}$

$$\mu(\mathrm{Int}(\bigcap_{i \in I} U_i)) = \inf_{i \in I} \mu(U_i).$$

Definition 1.7. A continuous, normalized, strongly non-degenerate, co-continuous valuation μ is called a *CC-valuation*.

Informally speaking, the strong non-degeneracy provides for non-zero contributions of compact elements and reasonable "pieces of space". The co-continuity provides for single non-compact elements and borders $B \setminus \mathrm{Int}(B)$ of "reasonable" sets $B \subseteq A$ to have zero measures.

"Reasonable" sets here are Aleksandrov open (i.e. upwardly closed) sets. Thus, it is possible to consider co-continuity as a method of dealing with non-discreteness of Scott topology. We follow here the remarkable definition of a discrete topology given by Aleksandrov: a topology is discrete if an intersection of arbitrary family of open sets is open (e.g. see [3]). Of course, if one assumes the T_1 separation axiom, then the Aleksandrov's definition implies that all sets are open — the trivial version of the definition, which is unfortunately the only one found in the most textbooks. In this sense, Aleksandrov topology of upwardly closed sets is discrete, but Scott topology is not. The further development of this viewpoint might prove to be fruitful.

We should also notice that since our valuations are bounded, they can be extended onto closed sets via formula $\mu(C) = \mu(A) - \mu(A \setminus C)$, and all definitions of this section can be expressed in the dual form. This gives us considerable benefits in applications of these valuations and also suggests that the generalization of these applications to the case of unbounded valuations is a non-trivial undertaking.

A bounded valuation μ can be uniquely extended to an additive measure defined on the ring of sets generated from the open sets by operations $\cap$, $\cup$, $\setminus$ [42]. We feel that it is useful to draft a possible construction here. We will denote the resulting additive measure also as μ.

First, consider all sets $U \cap C$, where U is open and C is closed. They form so-called semiring, which contains $\emptyset$, is closed under finite intersections, and possesses the following property: If $S = U \cap C$, $S_1 = U_1 \cap C_1$, $S_1 \subset S$, then there is a system of n non-intersecting sets $S_i = U_i \cap C_i$ (in fact, it is enough to take $n = 3$ here), such that $S = \bigcup_{i=1,\ldots,n} S_i$.

Then define $\mu(U \cap C) = \mu(U) - \mu(U \setminus C)$, using the fact that $U \setminus C = U \cap \overline{C}$ is an open set. Then observe that this definition is correct, i.e. if $U \cap C = U_1 \cap C_1$ then $\mu(U) - \mu(U \setminus C) = \mu(U_1) - \mu(U_1 \setminus C_1)$, and that the resulting measure is additive for the decomposition above: $\mu(S) = \mu(S_1) + \ldots + \mu(S_n)$. Then the unique extension to the ring of sets in question is the standard result of measure theory.

The issues of σ-additivity are not in the scope of this text (interested readers are referred to [31, 4]). We deal with the specific infinite systems of sets we need, and mainly focus on quite orthogonal conditions given to us by co-continuity of μ.

9.2 Examples

9.2.1 Valuations Based on Weights of Basic Elements

Consider a continuous dcpo A with a countable basis K. Assign the converging system of weights to basic elements: $w(k) > 0$, $\sum_{k \in K} w(k) = 1$. Define $\mu(U) = \sum_{k \in U} w(k)$. It is easy to see that μ is a continuous, normalized, strongly non-degenerate valuation.

However, μ is co-continuous if and only if all basic elements are compact (which is possible only if A is algebraic). To see why this is so, we undertake a small discourse, which will be useful for us later in this text.

First, observe that intersection of arbitrary system of Alexandrov open (i.e. upwardly closed) sets is Alexandrov open. Also it is a well-known fact that $\{y \mid y \gg x\}$ is Scott open in a continuous dcpo.

Lemma 9.2.1 (Border Lemma) *Consider an Alexandrov open set $B \subseteq A$. Then its interior in the Scott topology,* $\mathrm{Int}(B) = \{y \in A \mid \exists x \in B.\, x \ll y\}$. *Correspondingly, the border of B in the Scott topology,* $B \setminus \mathrm{Int}(B) = \{y \in B \mid \neg(\exists x \in B.\, x \ll y)\}$

Lemma 9.2.2 *If K is a basis of compact elements, then $\mu(U) = \sum_{k \in U} w(k)$, where $w(k) > 0$, $\sum_{k \in K} w(k) = 1$, defines a CC-valuation.*

Proof. (of co-continuity) Consider a filtered system of open sets $\{U_i,\, i \in I\}$. The set $B = \bigcap_{i \in I} U_i$ is Alexandrov open. Because $\{U_i\}$ is filtered, it is easy to see that $\sum_{k \in B} w(k) = \inf_{i \in I} \mu(U_i)$. Because basic elements of K are compact, $k \in K$, $k \in B \Rightarrow k \in \mathrm{Int}(B)$. Hence $\sum_{k \in B} w(k) = \sum_{k \in \mathrm{Int}(B)} w(k)$. $\square$

Lemma 9.2.3 *If K contains a non-compact element x, then $\mu(U) = \sum_{k \in U} w(k)$, where $w(k) > 0$, $\sum_{k \in K} w(k) = 1$ is not co-continuous.*

Proof. By the definition of basis, $x = \sqcup K_x$, where $K_x = \{k \in K \mid k \ll x\}$ is a directed set. For any k, consider $U_k = \{y \mid y \gg k\}$. Because K_x is directed, it is easy to see that $\{U_k,\, k \in K_x\}$ is filtered. Consider $B = \bigcap_{k \in K_x} U_k$. It is easy to see that $x \in B$.

Let us show that $x \notin \mathrm{Int}(B)$. If $x \in \mathrm{Int}(B)$ then there is $y \in B$ such that $y \ll x$ by the Border Lemma. Then $y \sqsubseteq k$ for some $k \in K_x$, and since B is upwardly closed, $k \in B$. This means that $\forall k' \in K_x.\, k' \ll k$ yielding $x \sqsubseteq k$ and $k \ll x$. Therefore $k = x$ and x is compact, which is a contradiction.

Thus,

$$x \notin \mathrm{Int}(B) \Rightarrow \sum_{k \in B} w(k) \geq \sum_{k \in \mathrm{Int}(B)} w(k) + w(x) > \sum_{k \in \mathrm{Int}(B)} w(k)$$

Since $\sum_{k \in B} w(k) = \inf_{k \in K_x} \mu(U_k)$ the Lemma is proved. □

9.2.2 A Vertical Segment of Real Line

Consider the segment [0, 1], $\sqsubseteq = \leq$. Define $\mu((x, 1]) = 1 - x$. Unfortunately, to ensure strong non-degeneracy we have to define $\mu([0, 1]) = 1 + \epsilon$, $\epsilon > 0$. This is the first hint that strong non-degeneracy might constitute too strong a restriction in many cases. In order to obtain a normalized valuation we have to consider $\mu'(U) = \mu(U)/(1 + \epsilon)$. The resulting μ' is a CC-valuation.

9.2.3 Interval Numbers

Consider the domain R^I of interval numbers belonging to the segment [0, A], where A is finite. Consider a Scott open $U \subseteq R^I$. Define $\mu_0(U)$ as the geometric area of U. This is a co-continuous valuation.

To insure strong non-degeneracy, we must concentrate some weight on segments $[[0, A],\ [0, 0]]$ and $[[0, A],\ [A, A]]$ and point $[0, A]$. In order to do that, we take positive $\epsilon, \epsilon_1, \epsilon_2$ and define $\mu(U) = \mu_0(U) + \epsilon_1 \cdot \mathit{Length}(U \cap [[0, A],\ [0, 0]]) + \epsilon_2 \cdot \mathit{Length}(U \cap [[0, A],\ [A, A]])$ when $U \neq R^I$, and $\mu(R^I) = 0.5 \cdot A^2 + (\epsilon_1 + \epsilon_2) \cdot A + \epsilon$.

Now we can obtain a CC-valuation μ' by defining $\mu'(U) = \mu(U)/\mu(R^I)$.

9.3 Constructing CC-valuations

In this section we build a CC-valuation for all continuous dcpo's with countable bases. The construction generalizes the one of Subsection 9.2.1. We are still going to assign weights, $w(k) > 0$, to compact elements. For non-compact basic elements we proceed as follows. We focus our attention on the pairs of non-compact basic elements, (k', k''), which do not have any compact elements between them, and call such elements *continuously connected*. We observe, that for every such pair we can construct a special kind of vertical chain, which "behaves like the vertical segment $[0, 1]$ of real line". We call such chain a *stick*. We assign weights, $v(k', k'') > 0$, to sticks as well, in such a way that the sum of all $w(k)$ and all $v(k', k'')$ is 1.

As in Subsection 9.2.1, compact elements k contribute $w(k)$ to $\mu(U)$, if $k \in U$. An intersection of the stick, associated with a continuously connected pair (k', k''), with an open set U "behaves as either $(q, 1]$ or $[q, 1]$", where $q \in [0, 1]$. Such stick contributes $(1 - q) \cdot v(k', k'')$ to $\mu(U)$. The resulting μ is the desired CC-valuation.

It is possible to associate a complete lattice homomorphism from the lattice of Scott open sets to $[0, 1]$ with every compact element and with every stick defined by basic continuously connected elements, k' and k''. Then, as suggested by Keimel [32], all these homomorphisms together can be thought of as an injective complete lattice homomorphism to $[0, 1]^J$. From this point of view, our construction of μ is the same as in [32].

Thus the discourse in this section yields the proof of the following:

Theorem 9.3.1 *For any continuous dcpo A with a countable basis, there is a CC-valuation μ on the system of its Scott open sets.*

9.3.1 Continuous Connectivity and Sticks

Definition 3.1. Two elements $x \ll y$ are called *continuously connected* if the set $\{k \in A \mid k \text{ is compact}, x \ll k \ll y\}$ is empty.

Remark: This implies that x and y are not compact.

Lemma 9.3.1 *If $x \ll y$ are continuously connected, then $\{z \mid x \ll z \ll y\}$ has cardinality of at least continuum.*

Proof. We use the well-known theorem on intermediate values that $x \ll y \Rightarrow \exists z \in A.\ x \ll z \ll y$ (see [28]). Applying this theorem again and again we build a countable system of elements between x and y as follows, using rational numbers as indices for intermediate elements:

$$x \ll a_{1/2} \ll y, \quad x \ll a_{1/4} \ll a_{1/2} \ll a_{3/4} \ll y, \ldots$$

All these elements are non-compact and hence non-equal. Now consider a directed set $\{a_i \mid i \leq r\}$, where r is a real number, $0 < r < 1$. Introduce $b_r = \sqcup\{a_i \mid i \leq r\}$. We prove that if $r < s$ then $b_r \ll b_s$, and also that $x \ll b_r \ll b_s \ll y$, thus obtaining the required cardinality. Indeed it is easy to find such n and numbers q_1, q_2, q_3, q_4, that

$$x \ll a_{q_1/2^n} \sqsubseteq b_r \sqsubseteq a_{q_2/2^n} \ll a_{q_3/2^n} \sqsubseteq b_s \ll a_{q_4/2^n} \ll y.$$

□

Definition 3.2. We call the set of continuum different non-compact elements $\{a_r \mid r \in (0,1)\}$ between continuously connected $x \ll y$, built in the proof above, such that $x \ll a_r \ll a_q \ll z \Leftrightarrow r < q$ a (vertical) *stick*.

9.3.2 Proof of Theorem 9.3.1

Consider a continuous dcpo A with a countable basis K. As discussed earlier, with every compact $k \in K$ we associate weight $w(k) > 0$, and with every continuously connected pair (k', k''), $k', k'' \in K$, we associate weight $v(k', k'') > 0$ and a stick $\{a_r^{k',k''} \mid r \in (0,1)\}$. Since K is countable, we can require $\sum w(k) + \sum v(k', k'') = 1$.

Whenever we have an upwardly closed (i.e. Alexandrov open) set U, for any stick $\{a_r^{k',k''} \mid r \in (0,1)\}$ there is a number $q_U^{k',k''} \in [0,1]$, such that $r < q_U^{k',k''} \Rightarrow a_r^{k',k''} \notin U$ and $q_U^{k',k''} < r \Rightarrow a_r^{k',k''} \in U$. In particular, for a Scott open set U define

$$\mu(U) = \sum_{k \in U \text{is compact}} w(k) \; + \sum_{k',k'' \in K \text{are continuously connected}} (1 - q_U^{k',k''}) \cdot v(k', k'')$$

It is easy to show that μ is a normalized valuation. The rest follows from the following Lemmas.

Lemma 9.3.2 *μ is continuous.*

Proof. Consider a directed system of open sets, $\{U_i, i \in I\}$, and $U = \bigcup_{i \in I} U_i$. We need to show that for any $\epsilon > 0$, there is such $U_i, i \in I$, that $\mu(U) - \mu(U_i) < \epsilon$.

Take enough (a finite number of) compact elements, $k_1, \ldots, k_n$, and continuously connected pairs of basic elements, $(k'_1, k''_1), \ldots, (k'_m, k''_m)$, so that $w(k_1) + \ldots + w(k_n) + v(k'_1, k''_1) + \ldots + v(k'_m, k''_m) > 1 - \epsilon/2$. For each $k_j \in U$, take $U_{i_j}, i_j \in I$, such that $k_j \in U_{i_j}$. For each (k'_j, k''_j), such that $q_U^{k'_j, k''_j} < 1$, take $U_{i'_j}, i'_j \in I$, such that $q_{U_{i'_j}}^{k'_j, k''_j} - q_U^{k'_j, k''_j} < \epsilon/(2m)$. An upper bound of these U_{i_j} and $U_{i'_j}$ is the desired U_i. $\square$

Lemma 9.3.3 *μ is strongly non-degenerate.*

Proof. Let U and V be Scott open subsets of A and $U \subset V$. Let us prove that $V \setminus U$ contains either a compact element or a stick between basic elements. Take $x \in V \setminus U$. If x is compact, then we are fine. Assume that x is not compact. We know that $x = \sqcup K_x$, $K_x = \{k \in K \mid k \ll x\}$ is directed set. Since V is open $\exists k \in K_x.\ k \in V$. Since $k \sqsubseteq x$ and $x \notin U$, $k \in V \setminus U$. If there is k' – compact, such that $k \ll k' \ll x$, we are fine, since $k' \in V \setminus U$. Otherwise, since any basis includes all compact elements, k and x are continuously connected.

Now, as in the theorem of intermediate values $x = \sqcup \widetilde{K}_x$, $\widetilde{K}_x = \{k' \in K \mid \exists k'' \in K.\ k' \ll k'' \ll x\}$ is directed set, thus $\exists k', k''.\ k \sqsubseteq k' \ll k'' \ll x$, thus (k, k'') yields the desired stick.

If $k \in V \setminus U$ and k is compact, then $\mu(V) - \mu(U) \geq w(k) > 0$. If the stick formed by (k, k') is in $V \setminus U$, then $\mu(V) - \mu(U) \geq v(k, k') > 0$. $\square$

Lemma 9.3.4 *μ is co-continuous.*

Proof. Recall the development in Subsection 9.2.1. Consider a filtered system of open sets $\{U_i,\ i \in I\}$. By Lemma 9.2.1 for $B = \bigcap_{i \in I} U_i$, $B \setminus \mathrm{Int}(B) = \{y \in B \mid \neg(\exists x \in B.\ x \ll y)\}$. Notice that $B \setminus \mathrm{Int}(B)$, in particular, does not contain compact elements. Another important point is that for any stick, $q_B^{k', k''} = q_{\mathrm{Int}(B)}^{k', k''}$.

The further development is essentially dual to the proof of Lemma 9.3.2. We need to show that for any $\epsilon > 0$, there is such $U_i, i \in I$, that $\mu(U_i) - \mu(\mathrm{Int}(B)) < \epsilon$.

Take enough (a finite number) of compact elements, $k_1, \ldots, k_n$, and continuously connected pairs of basic elements, $(k'_1, k''_1), \ldots, (k'_m, k''_m)$, so that $w(k_1) + \ldots + w(k_n) + v(k'_1, k''_1) + \ldots + v(k'_m, k''_m) > 1 - \epsilon/2$. For each $k_j \notin \mathrm{Int}(B)$, take $U_{i_j}, i_j \in I$, such that $k_j \notin U_{i_j}$. For each (k'_j, k''_j), such that $q^{k'_j,k''_j}_{\mathrm{Int}(B)} > 0$, take $U_{i'_j}, i'_j \in I$, such that $q^{k'_j,k''_j}_{\mathrm{Int}(B)} - q^{k'_j,k''_j}_{U_{i'_j}} < \epsilon/(2m)$. A lower bound of these U_{i_j} and $U_{i'_j}$ is the desired U_i. $\square$

It should be noted that Bob Flagg suggested and Klaus Keimel showed that Lemma 5.3 of [23] can be adapted to obtain a dual proof of existence of CC-valuations (see [22] for one presentation of this). Klaus Keimel also noted that one can consider all pairs k, k' of basic elements, such that $k \ll k'$, instead of considering just continuously connected pairs and compact elements.

9.4 Open Problems

The key open problems are related to fast computation of valuations and integrals and to canonical valuations on functional spaces and reflexive domains.

Since the similar problems exist for relaxed metrics we will return to them at the end of the next chapter.

Chapter 10

Relaxed and Partial Metrics

In this chapter we present our joint results with Joshua Scott [11, 12, 13]. We presume that the methods of denotational semantics allow us to obtain adequate descriptions of program behavior (e.g., see [53]). The term *domain* in this chapter denotes a directed complete partial order (*dcpo*) equipped with the Scott topology.

The traditional paradigm of denotational semantics states that all data types should be represented by domains and all computable functions should be represented by Scott continuous functions between domains. For the purposes of this chapter all *continuous* functions are Scott continuous.

Consider the typical setting in denotational semantics — a syntactic domain of programs, P, a semantic domain of meanings, A, and a continuous semantic function, $[\![\ ]\!] : P \to A$. The syntactic domain P (called a syntactic lattice in [53]) represents a data type of program parse trees, but we say colloquially that programs belong to P.

Assume that we have a domain representing distances, D, and a continuous generalized distance function, $\rho : A \times A \to D$. Assume that we can construct a *generalized metric topology*, $\mathcal{T}[\rho]$, on A via ρ. It would be reasonable to say that ρ reflects computational properties of A, if $\mathcal{T}[\rho]$ is the Scott topology on A.

Then $\rho([\![p_1]\!], [\![p_2]\!])$ would yield a computationally meaningful distance between programs p_1 and p_2. The continuous function ρ cannot possess all properties of ordinary metrics because we want $\mathcal{T}[\rho]$ to be non-Hausdorff.

10.1 Axiom $\rho(x, x) = 0$ Cannot Hold

Assume that there is an element $0 \in D$ representing the ordinary numerical 0. Let us show that $\forall x.\ \rho(x, x) = 0$ cannot be true under reasonable assumptions. We will see later that all other properties of ordinary metrics can be preserved at least for all continuous dcpo's with countable bases (see the next chapter).

It seems reasonable to assume that any reasonable construction of $\mathcal{T}[\rho]$ for any generalized distance function $\rho : A \times A \to D$, should satisfy the following axiom, regardless of whether the distance space D is a domain, or whether ρ is continuous:

Axiom 10.1.1 *For all $x, y \in A$, $\rho(x, y) = \rho(y, x) = 0$ implies that x and y share the same system of open sets, i.e. for all open sets $U \in \mathcal{T}[\rho]$, $x \in U$ iff $y \in U$.*

We assume this axiom for the rest of the chapter. A topology is called T_0, if different elements do not share the systems of open set.

Corollary 10.1.1 *If there are $x, y \in A$, such that $x \neq y$ and $\rho(x, y) = \rho(y, x) = 0$, then $\mathcal{T}[\rho]$ is not a T_0 topology.*

Let us return to our main case, where D is a domain and ρ is a continuous function.

Lemma 10.1.1 *Assume that there are at least two elements $x, y \in A$, such that $x \sqsubset_A y$. Assume that $\rho : A \times A \to D$ is a continuous function. If $\rho(x, x) = \rho(y, y) = d \in D$, then $\rho(x, y) = \rho(y, x) = d$.*

Proof. The continuity of ρ implies its monotonicity with respect to the both of its arguments. Then $x \sqsubset_A y$ implies $d = \rho(x, x) \sqsubseteq_D \rho(x, y) \sqsubseteq_D \rho(y, y) = d$. This yields $\rho(x, y) = d$, and, similarly, $\rho(y, x) = d$. $\square$

Then we can obtain the following simple, but important result.

Theorem 10.1.1 *Assume that there are at least two elements $x, y \in A$, such that $x \sqsubset_A y$. Assume that $\rho : A \times A \to D$ is a continuous generalized distance function and $\mathcal{T}[\rho]$ is a T_0 topology. Then the double equality $\rho(x,x) = \rho(y,y) = 0$ does not hold.*

Proof. By Lemma 10.1.1, $\rho(x,x) = \rho(y,y) = 0$ would imply $\rho(x,y) = \rho(y,x) = 0$. Then, by Corollary 10.1.1, $\mathcal{T}[\rho]$ would not be T_0, contradicting our assumptions. □

The topologies used in domain theory are usually T_0; in particular, the Scott topology is T_0. This justifies studying continuous generalized metrics ρ, such that $\rho(x,x) = 0$ is false for some x, more closely.

10.1.1 Intuition behind $\rho(x,x) \neq 0$

There are compelling intuitive reasons not to expect $\rho(x,x) = 0$, when x is not a maximal element of A. The computational intuition behind $\rho(x,y)$ is that the elements in question are actually x' and y', $x \sqsubseteq_A x', y \sqsubseteq_A y'$, but not all information is usually known about them. The correctness condition $\rho(x,y) \sqsubseteq_D \rho(x',y')$ is provided by the monotonicity of ρ.

In particular, even if $x = y$, this only means that we know the same information about x' and y', but this does not mean that $x' = y'$. Consider $x' \neq y'$, such that $x \sqsubset_A x'$ and $x \sqsubset_A y'$. Then $\rho(x,x) \sqsubseteq_D \rho(x',y')$ and $\rho(x,x) \sqsubseteq_D \rho(y',x')$, and at least one of $\rho(x',y')$ and $\rho(y',x')$ is non-zero, if we want ρ to yield a T_0 topology (we do not assume symmetry yet).

Example 10.1.1 Here is an important example — a continuous generalized distance on the domain of interval numbers R^I — $\rho : R^I \times R^I \to R^I$ (see Chapter 4 for the definition of R^I). Consider intervals $[a,b]$ and $[c,d]$ and set $S = \{|x'-y'| \mid a \leq x' \leq b,\ c \leq y' \leq d\}$. Define $\rho([a,b],[c,d]) = [\min S, \max S]$. In particular, $\rho([a,b],[a,b]) = [0, b-a] \neq [0,0]$, and $\rho([a,a],[b,b]) = [|a-b|, |a-b|]$.

10.2 Related Work

10.2.1 Quasi-Metrics

Quasi-metrics [50] and Kopperman-Flagg generalized distances [33] are asymmetric generalized distances. They satisfy the axiom $\rho(x, x) = 0$ and yield the Scott topology for various classes of domains via a construction satisfying Axiom 10.1.1.

Theorem 10.1.1 means that if one wishes to represent the distance spaces via domains, these asymmetric distances cannot be made continuous unless their nature is changed substantially.

The practice of representing all computable functions via continuous functions between domains suggests that quasi-metrics cannot, in general, be computed (see Section 10.8 for details in the effective setting).

10.2.2 Partial Metrics

Historically, partial metrics are the first generalized distances on domains for which the axiom $\rho(x, x) = 0$ does not hold. They were introduced by Matthews [39, 38] and further investigated by Vickers [54] and O'Neill [41].

Partial metrics satisfy a number of additional axioms in lieu of $\rho(x, x) = 0$ (see Section 10.5). Matthews and Vickers state that $\rho(x, x) \neq 0$ is caused by the fact, that x expresses a partially defined object. The most essential component of the central construction in this and the next chapter is a partial metric (Section 10.6).

10.2.3 Our Contribution

We build partial metrics yielding Scott topologies for a wider class of domains that was known before (see the next chapter). This class — all continuous dcpo's with countable bases — is sufficiently big to solve interesting domain equations and to define denotational semantics of at least sequential deterministic programming languages [53]. Actually, a recent careful analysis of papers [34, 39] together had shown that an even more general result followed easily from this papers taken together, however this fact remained unwritten folklore known only to a few people.

We introduce the notion of *relaxed metric* (Section 10.3), which maintains the intuitively clear requirement to reject axiom $\rho(x,x)=0$, but does not impose the specific axioms of partial metrics. We believe that the applicability of these specific axioms is more limited (Section 10.9.1).

We introduce the idea that a space of distances should be thought of as a data type in the context of denotational semantics and, thus, should be represented by a domain. We also introduce the requirement that distance functions should be computable and, thus, Scott continuous (the use of *continuous valuations* in [41] should be considered as a step in this direction).

These considerations lead to an understanding that relaxed metrics should map pairs of partial elements to *upper estimates* of some "ideal" distances, where the *distance domain of upper estimates*, R^-, is equipped with a dual informational order: $\sqsubseteq_{R^-} = \geq$. We also consider lower estimates of "ideal" distances, thus, introducing the *distance domain of interval numbers*, R^I. Continuous lower estimates are useful during actual computations of distances (Section 10.8) and for defining and computing an induced metric structure on the space of total elements (Theorem 10.6.2).

There is a comparison in [39] between partial metrics and alternative generalized distance structures such as quasi-metrics and weighted metrics [34]. We provide what we believe to be the strongest argument in favor of partial metrics so far — among all those alternatives only partial metrics can be thought of as Scott continuous, computable functions.

10.3 Relaxed Metrics

Consider distance domains in greater detail. It is conventional to think about distances as non-negative real numbers. When it comes to considering approximate information about reals, it is conventional to use some kind of *interval numbers*.

We follow both conventions in this text. The distance domain consists of pairs $\langle a,b\rangle$ (also denoted as $[a,b]$) of non-negative reals ($+\infty$ included), such that $a \leq b$.

Recall from the Chapter 4 that we denote this domain as R^I and that $[a,b] \sqsubseteq_{R^I} [c,d]$ iff $a \le c$ and $d \le b$.

Also recall that we can think about R^I as a subset of $R^+ \times R^-$, where $\sqsubseteq_{R^+} = \le$, $\sqsubseteq_{R^-} = \ge$, and both R^+ and R^- consist of non-negative reals and $+\infty$. We call R^+ a *domain of lower bounds*, and R^- a *domain of upper bounds*. Thus a distance function $\rho : A \times A \to R^I$ can be thought of as a pair of distance functions $\langle l, u \rangle$, $l : A \times A \to R^+$, $u : A \times A \to R^-$.

We think about $l(x,y)$ and $u(x,y)$ as, respectively, lower and upper bounds of some "ideal" distance $\sigma(x,y)$. We do not try to formalize the "ideal" distances, but we refer to them to motivate our axioms. There are good reasons to impose the triangle inequality, $u(x,z) \le u(x,y) + u(y,z)$. Assume that for our "ideal" distance, the triangle inequality, $\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$, holds. If $u(x,z) > u(x,y) + u(y,z)$, then $u(x,y) + u(y,z)$ gives a better upper estimate for $\sigma(x,z)$ than $u(x,z)$. This means that unless $u(x,z) \le u(x,y) + u(y,z)$, u could be easily improved and, hence, would be very imperfect.

This kind of reasoning is not valid for $l(x,z)$. In fact, there are reasonable situations, when $l(x,z) \ne 0$, but $l(x,y) = l(y,z) = 0$. E.g., consider Example 10.1.1 and take $x = [2,2], y = [2,3], z = [3,3]$.

Also only u plays a role in the subsequent definition of the relaxed metric topology, and the most important results remain true even if we take $l(x,y) = 0$. In the last case we sometimes take $D = R^-$ instead of $D = R^I$ making the distance domain look more like ordinary numbers (it is important to remember, that $\sqsubseteq_{R^-} = \ge$ and, hence, 0 is the largest element of R^-).

We also impose the symmetry axiom on the function ρ. The motivation here is that we presume our "ideal" distance to be symmetric, hence, we should be able to make symmetric upper and lower estimates.

We state a definition summarizing the discourse above:

Definition 10.3.1 A symmetric function $u : A \times A \to R^-$ is called a *relaxed metric* when it satisfies the triangle inequality. A symmetric function $\rho : A \times A \to R^I$ is called a *relaxed metric* when its upper part u is a relaxed metric.

10.4 Relaxed Metric Topology

An *open ball* with a center $x \in A$ and a real radius ϵ is defined as $B_{x,\epsilon} = \{y \in A \mid u(x, y) < \epsilon\}$. Notice that only upper bounds are used in this definition — the ball only includes those points y, about which we are *sure* that they are not too far from x.

We should formulate the notion of a relaxed metric open set more carefully than for ordinary metrics, because it is now possible to have a ball of a non-zero positive radius, which does not contain its own center.

Definition 10.4.1 A subset U of A is *relaxed metric open* if for any point $x \in U$, there is an $\epsilon > u(x, x)$ such that $B_{x,\epsilon} \subseteq U$.

It is easy to show that for a continuous relaxed metric on a dcpo all relaxed metric open sets are Scott open and form a topology.

10.5 Partial Metrics

The distances p with $p(x, x) \neq 0$ were first introduced by Matthews [39, 38]. They are known as *partial metrics* and obey the following axioms:

1. $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$.

2. $p(x, x) \leq p(x, y)$.

3. $p(x, y) = p(y, x)$.

4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The last axiom (due to Vickers [54]) implies the ordinary triangle inequality, since the distances are non-negative. O'Neill found it useful to introduce negative distances in [41], but this is avoided in the present work.

Whenever partial metrics are used to describe a partially ordered domain, a stronger form of the first two axioms is used: If $x \sqsubseteq y$ then $p(x,x) = p(x,y)$, otherwise $p(x,x) < p(x,y)$. We include the stronger form in the definition of partial metrics for the purposes of this work.

10.6 Central Construction

Here we construct continuous relaxed metrics yielding the Scott topology for all continuous Scott domains with countable bases. Our construction closely resembles one by O'Neill [41]. We also use valuations, but we consider continuous valuations on the powerset of the basis instead of continuous valuations on the domain itself. This allows us to handle a wider class of domains.

Consider a continuous Scott domain A with countable basis K. Enumerate elements of $K \backslash \{\perp_A\}$: $k_1, ..., k_i, ...$ Associate weights with all basic elements: $w(\perp_A) = 0$, and let $w(k_i)$ form a converging sequence of strictly positive weights. For convenience we agree that the sum of weights of all basic elements equals 1. For example, one might wish to consider $w(k_i) = 2^{-i}$ or $w(k_i) = \epsilon(1+\epsilon)^{-i}, \epsilon > 0$. Then for any $K_0 \subseteq K$, the weight of set K_0, $W(K_0) = \sum_{k \in K_0} w(k_0)$, is well defined and belongs to $[0,1]$.

We have several versions of function ρ. For most purposes it is enough to consider $u(x,y) = 1 - W(K_x \cap K_y)$ and $l(x,y) = 0$. Sometimes it is useful to consider a better lower bound function $l(x,y) = W(I_x \cup I_y)$, where $I_x = \{k \in K \mid k \sqcup x \text{ does not exist}\}$ for the computational purposes (Section 10.8).

Theorem 10.6.1 *The function u is a partial metric. The function ρ is a Scott continuous relaxed metric. The relaxed metric topology coincides with the Scott topology.*

If, in addition, we would like the next theorem to hold, we have to consider a different version of ρ with $u(x,y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y)$ and $l(x,y) = W(K_x \cap I_y) + W(K_y \cap I_x)$. The previous theorem still holds.

We introduce the notion of a *regular basis*. A set of maximal elements in A is denoted as *Total*(A) . We say that the basis K is *regular* if $\forall k \in K, x \in \mathit{Total}(A).\ k \sqsubseteq$

$x \Rightarrow k \ll x$. In particular, if K consists of compact elements, thus making A an algebraic Scott domain, K is regular.

Theorem 10.6.2 *Let K be a regular basis in A. Then for all x and y from $Total(A)$, $l(x,y) = u(x,y)$. Consider $\mu : Total(A) \times Total(A) \to \mathbf{R}$, $\mu(x,y) = l(x,y) = u(x,y)$. Then $(Total(A), \mu)$ is a metric space, and its metric topology is the subspace topology induced by the Scott topology on A.*

10.7 Proofs of Theorems 10.6.1 and 10.6.2

Here we prove relatively difficult parts of these theorems for the case when $u(x,y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y)$ and $l(x,y) = W(K_x \cap I_y) + W(K_y \cap I_x)$. Lemma 10.7.2 is needed for Theorem 10.6.2, and other lemmas are needed for Theorem 10.6.1.

Lemma 10.7.1 ((Correctness of lower bounds)) $l(x,y) \leq u(x,y)$.

Proof. Using $K_x \cap I_x = \emptyset$ we can rewrite u and l. $u(x,y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y) = W(\overline{U})$, where $U = (K_x \cap K_y) \cup (I_x \cap I_y)$. $l(x,y) = W(V)$, where $V = (K_x \cap I_y) \cup (K_y \cap I_x)$.

We want to show that $V \subseteq \overline{U}$, for which it is enough to show that $V \cap U = \emptyset$. We show that $(K_x \cap I_y) \cap U = \emptyset$. Then by symmetry the same will be true for $K_y \cap I_x$, and hence for V.

$(K_x \cap I_y) \cap U = (K_x \cap I_y \cap K_x \cap K_y) \cup (K_x \cap I_y \cap I_x \cap I_y)$. But $K_x \cap I_y \cap K_x \cap K_y \subseteq I_y \cap K_y = \emptyset$. Similarly, $K_x \cap I_y \cap I_x \cap I_y = \emptyset$. $\square$

Lemma 10.7.2 *If K is a regular basis and $x, y \in Total(A)$, then $l(x,y) = u(x,y)$.*

Proof. Using the notations of the previous proof we want to show that $\overline{U} \subseteq V$.

Let us show first that $K_x \cup I_x = K_y \cup I_y = K$. Consider $k \in K$. Since $x \in Total(A)$, if $k \notin I_x$, then $k \sqsubseteq x$. Now from the regularity of K we obtain $k \ll x$ and $k \in K_x$. Same for y.

Now, if $k \notin U$, then $k \notin K_x$ or $k \notin K_y$. Because of the symmetry it is enough to consider $k \notin K_x$. Then $k \in I_x$. Then, using $k \notin U$ once again, $k \notin I_y$. Then $k \in K_y$. So we have $k \in K_y \cap I_x \subseteq V$. $\square$

Lemma 10.7.3 ((Vickers-Matthews triangle inequality for upper bounds)) *:*

$u(x, z) \leq u(x, y) + u(y, z) - u(y, y)$.

Proof. We want to show $1 - W(K_x \cap K_z) - W(I_x \cap I_z) \leq 1 - W(K_x \cap K_y) - W(I_x \cap I_y) + 1 - W(K_y \cap K_z) - W(I_y \cap I_z) - 1 + W(K_y) + W(I_y)$. This is equivalent to $W(K_x \cap K_y) + W(K_y \cap K_z) + W(I_x \cap I_y) + W(I_y \cap I_z) \leq W(K_y) + W(I_y) + W(K_x \cap K_z) + W(I_x \cap I_z)$.

Notice that $W(K_x \cap K_y) + W(K_y \cap K_z) = W(K_x \cap K_y \cap K_z) + W(K_y \cap (K_x \cup K_z))$, and the similar formula holds for I's.

Then the result follows from the following simple facts: $W(K_x \cap K_y \cap K_z) \leq W(K_x \cap K_z)$, $W(K_y \cap (K_x \cup K_z)) \leq W(K_y)$, and the similar inequalities for I's. $\Box$

Lemma 10.7.4 *Function $\rho : A \times A \to R^I$ is continuous.*

Proof. Monotonicity of ρ is trivial.

Consider a directed set $B \subseteq A$ and some $z \in A$. We have to show that $\rho(z, \sqcup B) = \sqcup_{R^I} \{\rho(z, x) \mid x \in B\}$, which is equivalent to $u(z, \sqcup B) = \inf\{u(z, x) \mid x \in B\}$ and $l(z, \sqcup B) = \sup\{l(z, x) \mid x \in B\}$.

Rewriting this, we want to show that $W(K_z \cap K_{\sqcup B}) + W(I_z \cap I_{\sqcup B}) = \sup\{W(K_z \cap K_x) + W(I_z \cap I_x) \mid x \in B\}$ and $W(K_z \cap I_{\sqcup B}) + W(I_z \cap K_{\sqcup B}) = \sup\{W(K_z \cap I_x) + W(I_z \cap K_x) | \ x \in B\}$. Monotonicity considerations trivially yield both "$\geq$" inequalities, so it is enough to show "$\leq$" inequalities. In fact, we will show that for any sets $C \subseteq A$ and $D \subseteq A$, $W(C \cap K_{\sqcup B}) + W(D \cap I_{\sqcup B}) \leq \sup\{W(C \cap K_x) + W(D \cap I_x) \mid x \in B\}$ holds.

It is easy to show that $K_{\sqcup B} = \cup\{K_x \mid x \in B\}$ by showing first that the set $\cup\{K_x \mid x \in B\}$ is directed and $\sqcup B = \sqcup(\cup\{K_x \mid x \in B\})$. Let us prove that $I_{\sqcup B} = \cup\{I_x \mid x \in B\}$. "$\supseteq$" is trivial. Let us prove "$\subseteq$". Assume that $k \notin \cup\{I_x \mid x \in B\}$, i.e. $\forall x \in B.\ k \sqcup x$ exists. It is easy to see that because B is a directed set, $\{k \sqcup x \mid x \in B\}$ is also directed. Then $k \sqsubseteq \sqcup\{k \sqcup x \mid x \in B\} \sqsupseteq \sqcup B$, implying existence of $k \sqcup (\sqcup B)$ and, hence, $k \notin I_{\sqcup B}$.

Now consider enumerations of the countable or finite sets $K_{\sqcup B}$ and $I_{\sqcup B}$: $k_1, ..., k_n, ...$ and $k'_1, ..., k'_n, ...$, respectively. Define tail sums $S_n = w(k_n) + w(k_{n+1}) \ + ...$ and $S'_n = w(k'_n) + w(k'_{n+1}) + ...$ Observe that (S_n) and (S'_n) converge to 0.

Pick for every k_n some $x_n \in B$ such that $k_n \in K_{x_n}$. Pick for every k'_n some $x'_n \in B$ such that $k'_n \in I_{x'_n}$. Then using the directness of B, we can for any n pick such $y_n \in B$, that $x_1, ..., x_n, x'_1, ..., x'_n \sqsubseteq y_n$. Then $k_1, ..., k_n \in K_{y_n}$ and $k'_1, ..., k'_n \in I_{y_n}$. It is easy to see that $(W(C \cap K_{\sqcup B}) + W(D \cap I_{\sqcup B})) - (W(C \cap K_{y_n}) + W(D \cap I_{y_n})) = W(C \cap K_{\sqcup B}) - W(C \cap K_{y_n}) + W(D \cap I_{\sqcup B}) - W(D \cap I_{y_n}) < S_n + S'_n$, which can be made as small as we like. $\square$

Lemma 10.7.5 *If $B \subseteq A$ is Scott open then it is relaxed metric open.*

Proof. Consider $x \in B$. Because B is Scott open there is a basic element $k \in B$ such that $k \ll x$. We must find $\epsilon > 0$ such that $x \in B_{x,\epsilon} \subseteq B$. Let $\epsilon = u(x, x) + w(k)/2$. Clearly $x \in B_{x,\epsilon}$. We claim that $B_{x,\epsilon} \subseteq \{y | y \gg k\} \subseteq B$. Assume, by contradiction, that $y \gg k$ is false. Then $k \notin K_y$ and $k \in K_x$. Then $W(K_x \cap K_y) + W(I_x \cap I_y) + w(k) \leq W(K_x) + W(I_x)$. Then $u(x, y) - w(k) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y) - w(k) \geq 1 - W(K_x) - W(I_x) = u(x, x)$. Therefore $u(x, y) \geq u(x, x) + w(k)$ and thus $y \notin B_{x,\epsilon}$. $\square$

10.8 Computability and Continuity

Consider an effective algebraic Scott domain A and its computable element x. We recall that this means that K_x is recursively enumerable (Section 5.4). It is easy to show that in this case I_x must also be recursively enumerable. However, as we discussed in Section 5.4, one almost never should expect them to be recursive.

The actual computation of $\rho(x, y)$ goes as follows. Start with $[0, 1]$ and go along the recursive enumerations of K_x, K_y, I_x, and I_y. Whenever we discover that some k occurs in both K_x and K_y, or in both I_x and I_y, subtract $w(k)$ from the upper boundary. Whenever we discover that some k occurs in both K_x and I_y, or in both I_x and K_y, add $w(k)$ to the lower boundary. If this process continues long enough, $[l(x, y), u(x, y)]$ is approximated as well as desired.

However, there is no general way to compute a better lower estimate for $u(x, y)$ than $l(x, y)$, or to compute a better upper estimate for $l(x, y)$ than $u(x, y)$. Consequently, there is no general way to determine how close is the convergence process to the actual values of $l(x, y)$ and $u(x, y)$, except that we know that $u(x, y)$ is not less than the currently computed lower bound, and $l(x, y)$ is not greater than the currently computed upper bound. Of course, for large x and y this knowledge might provide a lot of information, and if the basis of our domain is regular, for total elements x and y this knowledge provides us with precise estimates — i.e. if the basis is regular, then the resulting classical metric on $Total(A)$ can be nicely computed.

The computational situation is very different with regard to quasi-metrics. Consider $u(x, y) = 1 - W(K_x \cap K_y)$ and $d(x, y) = u(x, y) - u(x, x) = W(K_x \setminus K_y)$. This is a quasi-metric in the style of [50, 33], and it yields a Scott topology [12, 13]. However, as discussed in [12, 13], typically $K_x \setminus K_y$ is not recursive. Moreover, one should not expect $K_x \setminus K_y$ or its complement to be recursively enumerable. This precludes us from building a generally applicable method computing $d(x, y)$ and illustrates that it is computationally incorrect to subtract one upper bound from another.

10.9 Axioms of Partial Metrics and Functoriality

10.9.1 Should the Axioms of Partial Metrics Hold?

Consider relaxed metric ρ and its upper part u. Should we expect function u to satisfy the axioms of partial metrics? Example 10.1.1 describes a natural relaxed metric on interval numbers, where u is not a partial metric. Function u gives a better upper estimate of the "ideal" distance between interval numbers, than the partial metric $p([a, b], [c, d]) = \max(b, d) - \min(a, c)$ described in [39]. For example, $u([0, 2], [1, 1]) = 1$, which is what one expects — if we know that one of the numbers is somewhere between 0 and 2, and another number equals 1, then we know that the distance between them is no greater than 1. However, $p([0, 2], [1, 1]) = p([0, 2], [0, 2]) = 2$.

Now we describe two situations when the axioms of partial metrics are justified. Consider again function $d(x, y) = u(x, y) - u(x, x)$ from Section 10.8. Vickers notes in [54] that the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ is equivalent to $u(x, z) \leq u(x, y) + u(y, z) - u(y, y)$, and $d(x, y) \geq 0$ is equivalent to $u(x, x) \leq u(x, y)$. This means that function u is a partial metric if and only if function d is a quasi-metric.

Another justification comes from the consideration of the proof of Lemma 10.7.3. Whenever the upper part $u(x, y)$ of a relaxed metric is based on *common information* shared by x and y yielding a *negative contribution* to the distance (we subtract the weight of common information from the universal distance 1 in this chapter), both $u(x, z) \leq u(x, y) + u(y, z) - u(y, y)$ and $u(x, x) \leq u(x, y)$ should hold. As we shall see in full generality in the next chapter, when $u(x, y) = 1 - \mu$(Common information between x and y), axioms of partial metrics hold.

However, to specify function u in Example 10.1.1, we use information about x and y, which cannot be thought of as common information shared by x and y. In such case we still expect a relaxed metric ρ, but its upper part u does not have to satisfy the axioms of partial metrics.

10.9.2 Partial Metrics and Functoriality

Here I would like to emphasize the utilitarian value of these axioms, i.e. certain useful properties which are easy to establish in the presence of these axioms. In all cases these considerations are closely related to the following series of *open problems*: What are the necessary and/or sufficient conditions for relaxed/partial metrics on various classes of domains in order for all Scott open sets to be relaxed metric open?

10.9.2.1 Open Balls Should be Open Sets

The strong Vickers–Matthews triangle inequality, $u(x, z) \leq u(x, y) + u(y, z) - u(y, y)$, is helpful in the proof that an open ball is a relaxed open set.

The proof goes as follows. Consider $y \in B_{x,\epsilon}$. We need to find $\delta > 0$, such that $B_{y,\delta+u(y,y)} \subseteq B_{x,\epsilon}$, i.e. $\forall z.\ u(y, z) \leq u(y, y) + \delta \Rightarrow u(x, z) < \epsilon$. Using $u(x, z) \leq u(x, y) + u(y, z) - u(y, y)$ and $\epsilon - u(x, y) > 0$ we take $\delta = (\epsilon - u(x, y))/2$ and obtain the desired result. This is closely related to the fact noted by Vickers, that for $d(x, y) = u(x, y) - u(x, x)$, $d(x, z) \leq d(x, y) + d(y, z) \Leftrightarrow u(x, z) \leq u(x, y) + u(y, z) - u(y, y)$. The ordinary triangle inequality $u(x, z) \leq u(x, y) + u(y, z)$ is not enough here, because we are looking for the ball around y with the radius $u(y, y) + \delta$ and not just δ.

10.9.2.2 Functoriality

One of the open issues in the field remains to develop a *functorial* approach to relaxed metrics. This approach should facilitate our abilities to both define canonical distances on domains and to compute these distances at a reasonable cost.

This method is to define reasonable relaxed metrics on some specific elementary domains, and then, given reasonable structures (relaxed metrics or, sometimes, measures) on domains A, B, define relaxed metric on domains $A \times B$, $A \to B$, etc. The natural candidates for relaxed metric on $A \times B$ and $A \to B$ are

$$\rho_{A\times B}(\langle x, y\rangle, \langle x', y'\rangle) = \alpha \cdot \rho_A(x, x') + \beta \cdot \rho_B(y, y'),\ \alpha, \beta > 0,\ \alpha + \beta = 1$$

$$\rho_{A\to B}(f, g) = \int_{x\in A} \rho_B(f(x), g(x))d\mu_x.$$

We also hope to be able to extend this approach to reflexive domains and to obtain some invariance properties of ρ in the process.

In all cases we would like to ensure that relaxed metric topology and Scott topology coincide. In particular, in order for Scott open sets to be relaxed metric open it is *necessary* (not sufficient) that $u(x, x) \geq u(x, y) \Rightarrow x \sqsubseteq y$.

However, for general relaxed metrics it is easy to come up with examples, when for both A and B $u_A(x, x) \geq u_B(x, y) \Rightarrow x \sqsubseteq y$ and $u_B(x', x') \geq u_B(x', y') \Rightarrow x' \sqsubseteq y'$, but $u_{A\times B}(\langle x, x'\rangle, \langle x, x'\rangle) \geq u_{A\times B}(\langle x, x'\rangle, \langle y, y'\rangle) \not\Rightarrow (x \sqsubseteq y \& x' \sqsubseteq y')$, and the similar situation takes place for $A \to B$. However, the strong form of the axiom of small self-distances, $u(x, x) \leq u(x, y)$ (namely, $x \sqsubseteq y \Rightarrow u(x, x) = u(x, y)$, $x \not\sqsubseteq y \Rightarrow u(x, x) < u(x, y)$) seems to eliminate these problems.

10.10 Applications and Open Problems

Let us briefly state where we stand with regard to the applications to programs. We are able to introduce relaxed metrics on a class of domains sufficiently large for practical applications in the spirit of [53].

For example, consider $X = [\![\texttt{while B do S}]\!]$, and the sequence of programs, $P_1 = \texttt{loop}; ...; P_N = \texttt{if B then S}; P_{N-1}\texttt{else skip endif}; ....$ Define $X_N = [\![P_N]\!]$. Typically $X_{N-1} \sqsubseteq X_N$ and $X = \sqcup X_N$ hold. We agreed that the distances between programs will be distances between their meanings. Assume that $M \leq N$. Then regardless of specific weights, $u(P_M, P_M) = u(P_M, P_N) = u(P_M, P)$, also $u(P_M, P_M) \geq u(P_N, P_N) \geq u(P, P)$, and $u(P, P) = \inf u(P_N, P_N)$. Of course, none of these distances has to be zero.

However, we do not know yet how to build relaxed distances so that not only nice convergence properties are true, but also that distances between particular pairs of programs "look right" — a notion, which is more difficult to formalize, than convergence. Also, we compute these distances via recursive enumeration now, and a more efficient scheme is needed.

Hence we think that the key open problems for relaxed distances are the same as for measures and integrals, namely interrelated problems of how to compute them quickly and how to define canonical valuations and relaxed distances on functional spaces and reflexive domains.

In Subsection 10.9.2 given a measure on A and a relaxed metric on B, we defined a functorial relaxed metric on $[A \to B]$. Hence the problem of functorial definitions for distances and measures are interrelated, and in order to approach higher-order functional spaces and reflexive domains for relaxed metrics, we should start with finding an appropriate construction for measures on $[A \to B]$.

It should be noted here that since we are going to distinguish functions topologically by integrating them over A, we need these measures to be based on strongly non-degenerate valuations.

Chapter 11

Obtaining Generalized Distances from Valuations

In this chapter we continue to present our joint results with Svetlana Shorina [15, 16].

In the previous chapter we introduced a construction of partial metrics based on the mechanism of *common information* between elements x and y bringing negative contribution to $u(x, y)$. This construction was based on assigning finite weights to basic elements and gave topologically meaningful relaxed metrics for all continuous Scott domains with countable bases. For such domains with regular bases (which included all algebraic Scott domains) this construction gave relaxed metrics which "behaved well" on the total elements of the domain.

Here we introduce a general method of defining partial and relaxed metrics via information about elements for all dcpo's via mechanism of μ*Info*-structures.

Then we show how to obtain such a μ*Info*-structure from a CC-valuation for any continuous Scott domain. Since we know, how to construct a CC-valuation for any continuous dcpo with countable basis, this method of constructing partial and relaxed metrics via CC-valuations and the resulting μ*Info*-structures works for all continuous Scott domains with countable bases.

11.1 Partial and Relaxed Metrics via Information

11.1.1 μ*Info*-structures

Assume that there is a set $\mathcal{I}$ representing information about elements of a dcpo A. We choose a ring, $\mathcal{M}(\mathcal{I})$, of admissible subsets of $\mathcal{I}$ and introduce a measure-like structure, μ, on $\mathcal{M}(\mathcal{I})$. We associate a set, $\mathit{Info}(x) \in \mathcal{M}(\mathcal{I})$, with every $x \in A$, and call $\mathit{Info}(x)$ a set of (positive) information about x. We also would like to consider negative information about x, $\mathit{Neginfo}(x) \in \mathcal{M}(\mathcal{I})$, — intuitively speaking, this is the information which cannot become true about x, when x is arbitrarily increased.

Definition 1.1. Given a dcpo A, the tuple of $(A, \mathcal{I}, \mathcal{M}(\mathcal{I}), \mu, \mathit{Info}, \mathit{Neginfo})$ is called a *μInfo-structure* on A, if $\mathcal{M}(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I})$ — a ring of subsets closed with respect to $\cap, \cup, \setminus$ and including $\emptyset$ and $\mathcal{I}$, $\mu : \mathcal{M}(\mathcal{I}) \to [0, 1]$, $\mathit{Info} : A \to \mathcal{M}(\mathcal{I})$, and $\mathit{Neginfo} : A \to \mathcal{M}(\mathcal{I})$, and the following axioms are satisfied:

1. **(VALUATION AXIOMS)**
   (a) $\mu(\mathcal{I}) = 1,\ \mu(\emptyset) = 0$;
   (b) $U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;
   (c) $\mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;
2. **(*Info* AXIOMS)**
   (a) $x \sqsubseteq y \Leftrightarrow \mathit{Info}(x) \subseteq \mathit{Info}(y)$;
   (b) $x \sqsubset y \Rightarrow \mathit{Info}(x) \subset \mathit{Info}(y)$;
3. **(*Neginfo* AXIOMS)**
   (a) $\mathit{Info}(x) \cap \mathit{Neginfo}(x) = \emptyset$;
   (b) $x \sqsubseteq y \Rightarrow \mathit{Neginfo}(x) \subseteq \mathit{Neginfo}(y)$;
4. **(STRONG RESPECT FOR TOTALITY)**
   $x \in \mathit{Total}(A) \Rightarrow \mathit{Info}(x) \cup \mathit{Neginfo}(x) = \mathcal{I}$;

5. **(SCOTT CONTINUITY OF THE INDUCED RELAXED METRIC)**

   if B is a directed subset of A and $y \in A$, then

   (a) $\mu(\mathit{Info}(\sqcup B) \cap \mathit{Info}(y)) = sup_{x \in B}(\mu(\mathit{Info}(x) \cap \mathit{Info}(y))$,

   (b) $\mu(\mathit{Info}(\sqcup B) \cap \mathit{Neginfo}(y)) = sup_{x \in B}(\mu(\mathit{Info}(x) \cap \mathit{Neginfo}(y))$,

   (c) $\mu(\mathit{Neginfo}(\sqcup B) \cap \mathit{Info}(y)) = sup_{x \in B}(\mu(\mathit{Neginfo}(x) \cap \mathit{Info}(y))$,

   (d) $\mu(\mathit{Neginfo}(\sqcup B) \cap \mathit{Neginfo}(y)) = sup_{x \in B}(\mu(\mathit{Neginfo}(x) \cap \mathit{Neginfo}(y))$;

6. **(SCOTT OPEN SETS ARE RELAXED METRIC OPEN)**

   for any (basic) Scott open set $U \subseteq A$ and $x \in U$, there is an $\epsilon > 0$, such that $\forall y \in A.\ \mu(\mathit{Info}(x)) - \mu(\mathit{Info}(x) \cap \mathit{Info}(y)) < \epsilon \Rightarrow y \in U$.

We will also consider *deficient* $\mu\mathit{Info}$-structures, when the **strong respect for totality** axiom is not imposed.

In terms of lattice theory, μ is a (normalized) valuation on a lattice $\mathcal{M}(\mathcal{I})$. The consideration of unbounded measures is beyond the scope of this work, and $\mu(\mathcal{I}) = 1$ is assumed for convenience. Axioms relating $\sqsubseteq$ and *Info* are in the spirit of information systems approach [48], although we are not considering any inference structure over $\mathcal{I}$ in this chapter.

The requirements for negative information are relatively weak, because it is quite natural to have $\forall x \in A.\ \mathit{Neginfo}(x) = \emptyset$ if A has the top element.

The axiom that for $x \in \mathit{Total}(A)$, $\mathit{Info}(x) \cup \mathit{Neginfo}(x) = \mathcal{I}$, is desirable because indeed, if some $i \in \mathit{Info}(x)$ does not belong to $\mathit{Info}(x)$ and x can not be further increased, then by our intuition behind $\mathit{Neginfo}(x)$, i should belong to $\mathit{Neginfo}(x)$. However, this axiom might be too strong and will be further discussed later.

The last two axioms are not quite satisfactory — they almost immediately imply the properties, after which they are named, but they are complicated and might be difficult to establish. We hope, that these axioms will be replaced by something more tractable in the future. One of the obstacles seems to be the fact in some valuable approaches (in particular, in this chapter) it is not correct that $x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ implies that $\mathit{Info}(\sqcup_{i \in \mathbf{N}} x_i) = \bigcup_{i \in \mathbf{N}} \mathit{Info}(x_i)$.

The nature of these set-theoretical representations, $\mathcal{I}$, of domains may vary: one can consider sets of tokens of information systems, powersets of domains bases, or powersets of domains themselves, custom-made sets for specific domains, etc. The approach via powersets of domain bases of the previous chapter can be thought of as a partial case of the approach via powersets of domains themselves adopted in this chapter.

11.1.2 Partial and Relaxed Metrics via μ*Info*-structures

Define the (upper estimate of the) distance between x and y from A as $u: A \times A \to \mathbf{R}^-$:

$$u(x,y) = 1 - \mu(\mathit{Info}(x) \cap \mathit{Info}(y)) - \mu(\mathit{Neginfo}(x) \cap \mathit{Neginfo}(y)).$$

I.e. the more information x and y have in common the smaller is the distance between them. However a partially defined element might not have too much information at all, so its self-distance $u(x,x) = 1 - \mu(\mathit{Info}(x)) - \mu(\mathit{Neginfo}(x))$ might be large.

It is possible to find information which will never made it into $\mathit{Info}(x) \cap \mathit{Info}(y)$ or $\mathit{Neginfo}(x) \cap \mathit{Neginfo}(y)$ even when x and y are arbitrarily increased. In particular, $\mathit{Info}(x) \cap \mathit{Neginfo}(y)$ and $\mathit{Info}(y) \cap \mathit{Neginfo}(x)$ represent such information. Then we can introduce the lower estimate of the distance $l: A \times A \to \mathbf{R}^+$:

$$l(x,y) = \mu(\mathit{Info}(x) \cap \mathit{Neginfo}(y)) + \mu(\mathit{Info}(y) \cap \mathit{Neginfo}(x)).$$

The proof of Lemma 10.7.1 is directly applicable and yields $(\mathit{Info}(x) \cap \mathit{Neginfo}(y)) \cup (\mathit{Info}(y) \cap \mathit{Neginfo}(x)) \subseteq \mathcal{I} \setminus ((\mathit{Info}(x) \cap \mathit{Info}(y)) \cup (\mathit{Neginfo}(x) \cap \mathit{Neginfo}(y)))$ implying $l(x,y) \leq u(x,y)$. Thus we can form an **induced relaxed metric**, $\rho: A \times A \to R^I$, $\rho = \langle l, u \rangle$, with a meaningful lower bound.

Now certain properties of partial and relaxed metrics can be established.

1. $u(x,x) \leq u(x,y)$.
   Indeed, axiom $U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$ implies $\mu(\mathit{Info}(x) \cap \mathit{Info}(y)) \leq \mu(\mathit{Info}(x))$, $\mu(\mathit{Neginfo}(x) \cap \mathit{Neginfo}(y)) \leq \mu(\mathit{Neginfo}(x))$, yielding $u(x,x) \leq u(x,y)$.

2. $u(x,z) \le u(x,y) + u(y,z) - u(y,y)$.

   The proof of Lemma 10.7.3 goes through.

   The symmetry $u(x,y) = u(y,x)$ and $l(x,y) = l(y,x)$ is obvious. Another helpful fact is

3. $x \sqsubseteq y \Rightarrow u(x,x) = u(x,y)$.

   We are using the axiom $x \sqsubseteq y \Rightarrow \mathit{Info}(x) \subseteq \mathit{Info}(y)$, which implies $\mathit{Info}(x) \cap \mathit{Info}(y) = \mathit{Info}(x)$, and the same for *Neginfo*.

4. $x \not\sqsubseteq y \Rightarrow u(x,x) < u(x,y)$.

   This follows from the following fact:

5. Any Scott open set is relaxed metric open.

   This, in turn, follows at once from the last axiom of μ*Info*-structure and $\mu(\mathit{Neginfo}(x) \cap \mathit{Neginfo}(y)) \le \mu(\mathit{Neginfo}(x))$.

6. Any relaxed metric open set is Scott open.

   This follows at once from the following property:

7. The induced relaxed metric is a Scott continuous function.

   This also follows immediately from the corresponding axiom of μ*Info*-structure.

Thus, the following theorem is proved (cf. Theorem 10.6.1). Observe, that the **strong respect for totality** axiom is NOT used in the proof.

Theorem 11.1.1 *Let A be a dcpo with a (possibly deficient) μInfo-strcuture and the induced relaxed metric $\rho = \langle l, u\rangle$. Then the function u is a partial metric, the function ρ is a Scott continuous relaxed metric, and the relaxed metric topology coincides with the Scott topology.*

Now let us assume that our μ*Info*-structure is not deficient.

Due to the axiom $\forall x \in \mathit{Total}(A).\ \mathit{Info}(x) \cup \mathit{Neginfo}(x) = \mathcal{I}$, the proof of Lemma 10.7.2 would go through, yielding

$$x, y \in \mathit{Total}(A) \Rightarrow l(x, y) = u(x, y)$$

and allowing to obtain the following theorem (cf. Theorem 10.6.2).

Theorem 11.1.2 *Let A be a dcpo with a μInfo-strcuture and the induced relaxed metric $\rho = \langle l, u\rangle$. Then for all x and y from $\mathit{Total}(A)$, $l(x, y) = u(x, y)$. Consider $\mu : \mathit{Total}(A) \times \mathit{Total}(A) \to \mathbf{R}$, $\mu(x, y) = l(x, y) = u(x, y)$. Then $(\mathit{Total}(A), \mu)$ is a metric space, and its metric topology is the subspace topology induced by the Scott topology on A.*

However, in the previous chapter $x \in \mathit{Total}(A) \Rightarrow \mathit{Info}(x) \cup \mathit{Neginfo}(x) = \mathcal{I}$ holds under an awkward condition of the regularity of the basis. While bases of algebraic Scott domains and of continuous lattices can be made regular, there are important continuous Scott domains, which cannot be given regular bases. In particular, in $\mathbf{R}^I$ no element, except for $\perp$, satisfies the condition of regularity, hence a regular basis cannot be provided for $\mathbf{R}^I$.

The achievement of the construction to be described in the Section 11.1.4 is that by removing the reliance on the weights of non-compact basic elements, it eliminates the regularity requirement and implies $x \in \mathit{Total}(A) \Rightarrow \mathit{Info}(x) \cup \mathit{Neginfo}(x) = \mathcal{I}$ for all continuous Scott domains equipped with a CC-valuation (which is built above for all continuous Scott domains with countable bases) where $\mathit{Info}(x)$ and $\mathit{Neginfo}(x)$ are as described below in the Subsection 11.1.4.

However, it might still be fruitful to consider replacing the axiom $\forall x \in \mathit{Total}(A).\ \mathit{Info}(x) \cup \mathit{Neginfo}(x) = \mathcal{I}$ by something like $\forall x \in \mathit{Total}(A).\ \mu(\mathcal{I} \setminus (\mathit{Info}(x) \cup \mathit{Neginfo}(x))) = 0$.

11.1.3 A Previously Known Construction

Here we recall a construction from the previous chapter based on a generally non-co-continuous valuation of Subsection 9.2.1. We will reformulate it in our terms of *μInfo-*

structures. In the previous chapter it was natural to think that $\mathcal{I} = K$. Here we reformulate that construction in terms of $\mathcal{I} = A$, thus abandoning the condition $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$ altogether.

Define $I_x = \{y \in A \mid \{x, y\} \text{ is unbounded}\}$, $P_x = \{y \in A \mid y \ll x\}$ (cf. $I_x = \{k \in K \mid \{k, x\} \text{ is unbounded}\}$, $K_x = \{k \in K \mid k \ll x\}$ in [11]).

Define $Info(x) = P_x$, $Neginfo(x) = I_x$. Consider a valuation μ of Subsection 9.2.1: for any $S \subset \mathcal{I} = A$, $\mu(S) = \sum_{k \in S \cap K} w(k)$. μ is a continuous strongly non-degenerate valuation, but it is not co-continuous unless K consists only of compact elements.

Because of this we cannot replace an inconvenient definition of $Info(x) = P_x$ by $Info(x) = C_x = \{y \in A \mid y \sqsubseteq x\}$ (which would restore the condition $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = A$) as $\mu(C_k)$ would not be equal to $\sup_{k' \ll k} \mu(C_{k'})$ if k is a non-compact basic element, leading to the non-continuity of the partial metric $u(x, y)$.

Also the reliance on countable system of finite weights excludes such natural partial metrics as metric $u : \mathbf{R}^-_{[0,1]} \times \mathbf{R}^-_{[0,1]} \to \mathbf{R}^-$, where $\mathbf{R}^-_{[0,1]}$ is the set [0, 1] equipped with the dual partial order $\sqsubseteq = \geq$, and $u(x, y) = max(x, y)$. We rectify all these problems in the next Subsection.

11.1.4 Partial and Relaxed Metrics via CC-valuations for Continuous Scott Domains

Assume that there is a CC-valuation $\mu(U)$ on Scott open sets of a continuous Scott domain A. Then it uniquely extends to an additive measure μ on the ring of sets generated by the system of open sets. Define $\mathcal{I} = A$, $Info(x) = C_x$, $Neginfo(x) = I_x$. It is easy to see that valuation, *Info*, and *Neginfo* axioms of μ*Info*-structure hold. We have $x \in Total(A) \Rightarrow C_x \cup I_x = A$. Thus we only need to establish μ*Info*-structure axioms of **Scott continuity of the induced relaxed metric** and **Scott open sets are relaxed metric open** in order to prove theorems 11.1.1 and 11.1.2 for our induced relaxed metric, $u(x, y) = 1 - \mu(C_x \cap C_y) - \mu(I_x \cap I_y)$, $l(x, y) = \mu(C_x \cap I_y) + \mu(C_y \cap I_x)$. These axioms are established by the Lemmas below.

You will also see that for such bare-bones partial metrics, as $u(x, y) = 1 - \mu(C_x \cap C_y)$, which are nevertheless quite sufficient for topological purposes and for domains with $\top$, only *co-continuity* of valuations matters, continuity is not important.

Observe also that since the construction in Section 9.2.1 does form a CC-valuation for algebraic Scott domains with bases of compact elements, the construction in the previous chapter can be considered as a partial case of our current construction if the basis does not contain non-compact elements.

Lemma 11.1.1 *Assume that μ is a co-continuous valuation and B is a directed subset of A. Then $\mu(C_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(C_x \cap Q))$, where Q is a closed or open subset of A.*

Remark: Note that continuity of μ is not required here.

Proof. Part A: Let Q be a closed subset of A. Then $(C_x \cap Q,\ x \in B)$ form a directed set of closed sets. We need to show that $\overline{(\bigcup_{x \in B} C_x) \cap Q} = C_{\sqcup B} \cap Q$, then the result will follow by co-continuity. "$\subseteq$" is trivial. Let us proof "$\supseteq$". Consider $x \in C_{\sqcup B} \cap Q$. $x = \sqcup P_x$; $P_x = \{y \in A \,|\, y \ll x\}$ is directed set and $P_x \subseteq C_{\sqcup B} \cap Q$. However, for $\forall y \in P_x$, since $y \ll x$ and $x \sqsubseteq \sqcup B$, $\exists z \in B.\ y \sqsubseteq z$, hence $y \in C_z \cap Q$. Hence $P_x \subseteq \bigcup_{z \in B}(C_z \cap Q) = (\bigcup_{z \in B} C_z) \cap Q$, and $x \in \overline{P_x}$ implies $x \in \overline{(\bigcup_{z \in B} C_z) \cap Q}$.

Part B: If Q is an open set, then $\mu(C_x \cap Q) = \mu(C_x) - \mu(C_x \setminus Q)$; $\mu(C_{\sqcup B} \cap Q) = \mu(C_{\sqcup B}) - \mu(C_{\sqcup B} \setminus Q)$. The non-trivial part is to show $\mu(C_{\sqcup B} \cap Q) \leq \sup_{x \in B} \mu(C_x \cap Q)$. This follows from $\mu(C_{\sqcup B}) = \sup_{x \in B} \mu(C_x)$ by Part A and $\mu(C_{\sqcup B} \setminus Q) \geq \sup_{x \in B} \mu(C_x \setminus Q)$ by monotonicity of μ. $\square$

Lemma 11.1.2 *Assume that μ is a continuous valuation and B is a directed subset of A. Then $\mu(I_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(I_x \cap Q))$, were Q is an open or closed subset of A.*

Remark: Co-continuity is not needed here.

Proof. During the proof of Lemma 10.7.4 it was established that $I_{\sqcup B} = \bigcup_{x \in B} I_x$ (think about $k \in A$ and observe that that proof goes through).

Part A: If Q is an open set, $I_{\sqcup B} \cap Q = (\bigcup_{x \in B} I_x) \cap Q = \bigcup_{x \in B}(I_x \cap Q)$ and $\mu(I_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(I_x \cap Q))$ is a direct consequence of continuity of μ.

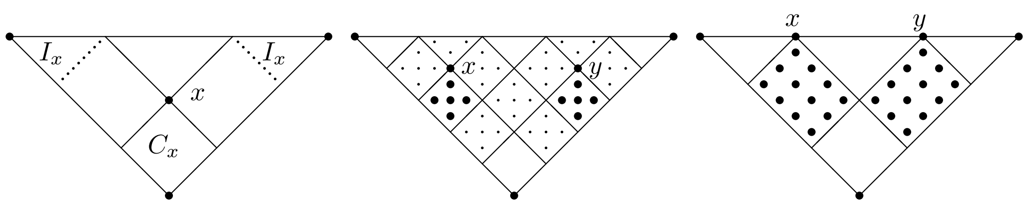
Part B: If Q is closed, let us think about $\mu(I \cap Q)$ as $\mu(I) - \mu(I \setminus Q)$, where both I and $I \setminus Q$ are open sets. The rest is similar to Part B of the proof of the previous Lemma. $\square$

Lemma 11.1.3 *Assume that μ is a strongly non-degenerate valuation. Then the μInfo-structure axiom* **Scott open sets are relaxed metric open** *holds.*

Remark: Neither continuity, nor co-continuity are required, and even strong non-degeneracy condition can be probably be made weaker (see the next section).

Proof. Consider a Scott open set U and $x \in U$. We would like to find $\delta > 0$, such that $\mu(C_x) - \mu(C_x \cap C_y) < \delta \Rightarrow y \in U$ (and hence, $B_{x,\delta+u(x,x)} \subseteq U$). Using $x = \sqcup P_x$, where $P_x = \{y \mid y \ll x\}$ is a directed set, obtain $\exists y \ll x.\ y \in U$ (this includes the case of compact x, $y = x$). Consider a Scott open set $\{z \in A \mid y \ll z\} \ni x$ and a Scott open set $A \setminus C_x$. Then $x \notin (A \setminus C_x)$, hence $(A \setminus C_x) \cup \{z \in A \mid y \ll z\} \supset (A \setminus C_x)$ and by strong non-degeneracy $\mu((A \setminus C_x) \cup \{z \in A \mid y \ll z\}) - \mu(A \setminus C_x) = \epsilon > 0$. We take $\delta = \epsilon$ and prove that $\mu(C_x) - \mu(C_x \cap C_z) < \delta$ implies $y \sqsubseteq z$, and thus $z \in U$. Indeed, assume $y \not\sqsubseteq z$. Then we have that $y \notin C_z$, thus $\{z \mid y \ll z\} \subseteq (A \setminus C_z)$, thus $\mu((A \setminus C_x) \cup (A \setminus C_z)) \geq \mu(A \setminus C_x) + \epsilon$, implying $\mu(C_x) - \mu(C_x \cap C_z) \geq \delta$. $\square$

It is helpful to visualize the situation of this section via the following pictures. They show sets C_x and I_x, show the "area" accounting for $l(x, y)$ as shaded with large dots, the "area" accounting for $u(x, y)$ as shaded with all dots (large and small), and show why $l(x, y) = u(x, y)$, when $x, y \in Total(A)$.



11.2 Examples and Non-degeneracy Issues

In this section we show some examples of "nice" partial metrics, based on valuations for vertical and interval domains of real numbers. Some of these valuations are strongly non-degenerate, while others are not, yet all examples are quite natural.

Consider the example from Subsection 9.2.2. The partial metric, based on the strongly non-degenerate valuation μ' of that example would be $u'(x,y) = (1 - \min(x,y))/(1+\epsilon)$, if $x,y > 0$, and $u'(x,y) = 1$, if x or y equals to 0. However, another nice valuation, μ'', can be defined on the basis of μ of Subsection 9.2.2: $\mu''((x,1]) = \mu((x,1]) = 1 - x$, $\mu''([0,1]) = 1$. μ'' is not strongly non-degenerate, however it yields nice partial metric $u''(x,y) = 1 - \min(x,y)$, yielding the Scott topology.

Now we consider several valuations and distances on the domain of interval numbers located within the segment $[0,1]$. This domain can be thought of as a triangle of pairs $\langle x,y \rangle$, $0 \leq x \leq y \leq 1$. Various valuations can either be concentrated on $0 < x \leq y < 1$, or on $x = 0$, $0 \leq y \leq 1$ and $y = 1$, $0 \leq x \leq 1$, or, to insure non-degeneracy, on the both of these areas with an extra weight at $\langle 0,1 \rangle$.

The classical partial metric $u([x,y],[x',y']) = \max(y,y') - \min(x,x')$ results from the valuation accumulated at $x = 0$, $0 \leq y \leq 1$, and $y = 0$, $0 \leq x \leq 1$, namely $\mu(U) = (\mathit{Length}(\{x = 0,\, 0 \leq y \leq 1\} \cap U) + \mathit{Length}(\{y = 1,\, 0 \leq x \leq 1\} \cap U))/2$. Partial metrics denerated by strongly non-degenerate valuations contain quadratic expressions.

It is our current feeling, that instead of trying to formalize weaker non-degeneracy conditions, it is often more fruitful to change a μ*Info*-structure. In particular, one can build a μ*Info*-structure based on $\mathcal{I} = [0,1] \times [0,1]$ in the situation described above.

Chapter 12

Negative Information and Tolerances

In this chapter we present our later joint results with Svetlana Shorina [17, 14]. In the previous chapter we showed how to obtain such a μ*Info*-structure from a CC-valuation for any continuous Scott domain. Certain pathologies in the behavior of $I_x = \{y \in A \mid \{x, y\} \text{ is unbounded}\}$ precluded us from extending this method beyond bounded complete domains.

In this chapter, we obtain meaningful Scott continuous relaxed metrics for continuous dcpo's by replacing I_x with $\mathit{Neginfo}(x) = \mathrm{Int}(J_x)$, where $J_x = \{y \in A \mid x \in \mathrm{Int}(I_y)\}$. This result can be understood in terms of interplay between *negation duality* and *Stone duality*. Since we know how to construct a CC-valuation for any continuous dcpo with countable basis, this method of constructing partial and relaxed metrics via CC-valuations and the resulting μ*Info*-structures works for all continuous dcpo's with countable bases.

Escardo [21, 52] defined a topological space A to be *weakly Hausdorff*, if its consistency relation is closed. The consistency relation is given by formula $x \uparrow y = \{\langle x, y\rangle \mid \exists z \in A.\, x \sqsubseteq z \& y \sqsubseteq z\}$, where $\sqsubseteq$ is the specialization order of A.

A continuous dcpo with Scott topology is weakly Hausdorff, if and only if negative information I_x is observable, e.g. Scott open, due to the fact that the relation $\{\langle x, y\rangle \mid y \in I_x\}$ is exactly the complement of the consistency relation.

The technique of considering $\mathit{Neginfo}(x) = \mathrm{Int}(J_x)$ works especially well, when a continuous dcpo A satisfies the *Lawson condition* — the relative Scott and Lawson topologies on $\mathit{Total}(A)$ are equal. We obtain that the Lawson condition is equivalent to the property $\forall x \in \mathit{Total}(A).\, I_x = J_x$. This will imply that when the Lawson condition holds, an induced classical metric on $\mathit{Total}(A)$ results.

Since the Lawson condition is equivalent to the formula $\forall x \in \mathit{Total}(A).\, I_x = J_x$, which is a weakening of $I = J$, which is, in turn, equivalent to A being weakly Hausdorff, we call the spaces satisfying the Lawson condition *very weakly Hausdorff*.

12.1 Tolerances and a Smyth Conjecture

Recently Mike Smyth [52] and Julian Webster [58] advanced the approach in which tolerance is considered not as an alternative to standard topology, but as a structure complementary to topology. In particular, it seems to be fruitful to equip Scott domains with tolerances.

Sections 12.6 and 12.7 make a small contribution to this emerging theory.

Smyth [52] defines a tolerance as a reflexive symmetric relation following Poincare and Zeeman. He defines a topological tolerance space as a topological space equipped with a tolerance relation closed in the product topology.

For a weakly Hausdorff space, $\uparrow$ is the least closed tolerance.

The set $I_x = \{y \in A \mid \{x, y\} \text{ is unbounded}\}$ is an observable continuous representation of negative information about $x \in A$ for a weakly Hausdorff continuous dcpo A with the Scott topology. We will see in Section 12.4 that, when A is not weakly Hausdorff the largest continuous approximation of I_x is represented by $J_x = \{y \in A \mid x \in \mathrm{Int}(I_y)\}$, and the largest observable continuous representation of I_x is represented by $J'_x = \mathrm{Int}(J_x)$.

Smyth conjectured, that J or J' is closely related to the least symmetric closed tolerance on A. In this chapter we establish that, indeed, $\{\langle x, y\rangle \mid y \in J'_x\}$ is the complement of this tolerance.

We also establish a relationship between this tolerance and lower bounds of relaxed metrics on A.

12.2 Overview of the Chapter

12.2.1 Negation Duality and Problems with I_x

Certain pathologies in the behavior of I_x do not allow us to use the formula $\mathit{Neginfo}(x) = I_x = \{y \in A \,|\, \{x, y\} \text{ is unbounded}\}$ when A is not bounded complete. It is convenient to use *negation duality*, $x \in I_y \Leftrightarrow y \in I_x$, when analyzing the behavior of I_x. Specifically, one can consider directed sets $B \subseteq A$, and by considering $x = \sqcup B$ and $x = b \in B$, obtain the following Lemma:

Lemma 12.2.1 *For any dcpo A, $(\forall y \in A.\, I_y$ is Scott open$) \Leftrightarrow (\forall B \subseteq A.\, B$ is directed $\Rightarrow I_{\sqcup B} = \bigcup_{b \in B} I_b)$.*

Proof. $\Rightarrow$. Consider $y \in I_{\sqcup B}$. $y \in I_{\sqcup B} \Leftrightarrow \sqcup B \in I_y$. If I_y is Scott open, $\exists b \in B.\, b \in I_y$, and $b \in I_y \Leftrightarrow y \in I_b$.

$\Leftarrow$. Consider directed B, such that $\sqcup B \in I_y$. $\sqcup B \in I_y \Leftrightarrow y \in I_{\sqcup B}$. Condition $I_{\sqcup B} = \bigcup_{b \in B} I_b$ implies that $\exists b \in B.\, y \in I_b$, and $y \in I_b \Leftrightarrow b \in I_y$. $\square$

One can informally restate this, by saying that all I_y are (Scott) observable [51], if and only if I is a Scott continuous function $A \to \mathcal{P}(A)$.

We will see that for some continuous dcpo's certain I_y's are not Scott open. This would be possible to tolerate, since, as we will see later, co-continuity of valuations allows to extend those valuations to Alexandrov open sets, and any I_y is still Alexandrov open. However, the corresponding breakdown of equality $I_{\sqcup B} = \bigcup_{b \in B} I_b$ for directed sets B cannot be tolerated, because it tends to lead to the loss of Scott continuity for the resulting relaxed metrics.

We could have replaced I_x with $\mathrm{Int}(I_x)$ (here and everywhere in this chapter, interiors are being taken in Scott topology), to rectify the problem of I_x being not Scott open, but this does nothing to fix the broken continuity property. Somewhat surprisingly, the *negation duality* helps us to resolve this problem.

12.2.2 Negation Duality and Stone Duality

We consider equality $I_x = \{y \in A \mid x \in I_y\}$, which is just another way to state the negation duality. Then, instead of taking $\mathrm{Int}(I_x)$, we consider $J_x = \{y \in A \mid x \in \mathrm{Int}(I_y)\}$. Then the continuity property for the resulting relaxed metric will be restored. It is going to be technically convenient to replace I_x not with J_x, but with $\mathrm{Int}(J_x)$, but we do not think that this feature is principal.

We will see that the reasons for J_x to work in this situation can be best understood in terms of Stone duality. In particular, the map $y \mapsto \mathrm{Int}(I_y)$ gives rise to a map of the generators $\Uparrow\{y\}$ of the free frame of Scott open sets of the powerset of A (ordered by inclusion, the original ordering on A is ignored) to the frame of open sets of the domain A. Function $x \mapsto J_x$ turns out to be a Scott continuous function $A \to \mathcal{P}(A)$, which is dual to the function $\mathcal{O}(\mathcal{P}(A)) \to \mathcal{O}(A)$ obtained by extending map $\Uparrow\{y\} \mapsto \mathrm{Int}(I_y)$ onto the frame $\mathcal{O}(\mathcal{P}(A))$. We should emphasize here, that what is going on in this chapter is a rather subtle interplay of two *different* dualities — negation duality and Stone duality — none of which seems to be reducible to another.

We will also see, that the function $x \mapsto J_x$ is the largest "negative" Scott continuous function $A \to \mathcal{P}(A)$, where the powerset of A, $\mathcal{P}(A)$, is ordered by the set-theoretic inclusion. This means that if $f : A \to \mathcal{P}(A)$ is another Scott continuous function, such that for all $x \in A$, $f(x) \subseteq I_x$, then for all $x \in A$, $f(x) \subseteq J_x$.

12.2.3 Lawson Condition

In general we only get deficient μ*Info*-structures via the use of *Neginfo*$(x) = J_x$ or *Neginfo*$(x) = \mathrm{Int}(J_x)$. Namely, the totality property does not hold in general, and thus we still do not get an induced metric on *Total*(A). However, in this situation it helps to impose the *Lawson condition*, that relative Scott and Lawson topologies on *Total*(A) are equal.

The Lawson condition was introduced in [37], and is widely used lately. As

Lawson writes in the Introduction to [37], "this turns out to be a very fruitful notion that permit great generality, but at the same time permits the derivation of many important structure results". This condition is now a standard part of the notion of a *computational model* for a topological space [22].

In this chapter, we present two equivalent formulations of the Lawson condition: $\forall x \in A.\, I_x \cap \mathit{Total}(A) = \mathrm{Int}(I_x) \cap \mathit{Total}(A)$ and $\forall x \in \mathit{Total}(A).\, J_x = I_x$.

The second of these equivalent formulations implies that if the Lawson condition holds for the domain A and, hence, for any maximal element x, $\mathrm{Int}(J_x) = J_x = I_x$, then the totality property, $C_x \cup \mathit{Neginfo}(x) = A$, holds, and the induced metric on $\mathit{Total}(A)$ results. Of course, the resulting metric topology is the same as relative Scott and Lawson topologies on $\mathit{Total}(A)$.

12.2.4 Polish Spaces

Lawson has shown in [37], that for every continuous dcpo A with countable basis, Lawson condition implies that $\mathit{Total}(A)$ is a Polish space, i.e. that it is homeomorphic to a complete, separable metric space.

However, since completeness of metric spaces is not a topological invariant, this does not mean that metrics obtained by our present methods have to be complete. Indeed, our present methods, based on assignment of converging systems of finite weights for the case of algebraic domains, yield a non-complete metric on $\mathit{Total}(E')$ for the domain E' of Section 12.5.

This raises a lot of open questions, ranging from the question of when our construction yields a complete metric space to the question of whether methods of Edalat and Heckmann, used to approximate complete metric spaces (see [27] for the variant of their approach using partial metrics and, thus, closest to our methods), can be extended to describe certain non-complete metric spaces, like an open interval of the real line or set $\{1/2, 1/4, 1/8, \ldots\}$.

12.2.5 Historical Remarks

Bob Flagg noted, that our results from Section 12.3 can be generalized to continuous dcpo's with compact Lawson topology. Mike Smyth observed that this is a corollary of the following two facts. The first fact is that the conditions that all I_x are Scott open, that $I = J$, and that the continuous dcpo is weakly Hausdorff are equivalent. The second fact is that continuous dcpo's with compact Lawson topology are weakly Hausdorff [52].

Since $I = J$ is equivalent to the space in question being weakly Hausdorff, and since the Lawson condition can be reformulated as $I_x = J_x$ for maximal x, we can offer an alternative name for the spaces, satisfying the Lawson condition — *very weakly Hausdorff spaces.*

Using this terminology, we can say that one of the discoveries of this chapter is that continuous dcpo's do not have to be weakly Hausdorff to be satisfactorily relaxed metrizable by our methods, but it is enough to require that they be very weakly Hausdorff.

12.2.6 Structure of the Chapter

In Section 12.3 we show that the old formulas, based on I_x, work for the class of *coherently continuous* dcpo's with countable basis, because I_x's are Scott open for this class of domains.

In Section 12.4 we analyze the pathologies of behavior of I_x on a specific example. We then study the properties of J_x, which serves as a replacement for I_x, and explain those properties from the viewpoint of Stone duality.

In Section 12.5 we find equivalent formulations of the Lawson condition and use these formulations to establish the totality property of the resulting $\mu Info$-structures with its ramifications for the induced metrics on $Total(A)$.

In Section 12.6 we talk about tolerances and prove the Smyth Conjecture.

In Section 12.7 we establish that for the relaxed metrics defined above, $x \not\sim y$ if and only if $l(x, y) \neq 0$, and build a continuous family of tolerances.

12.3 When I_x Behaves Well

In this section we study cases, when for all $x \in A$, I_x is Scott open, or, equivalently, when for all directed $B \subseteq A$, $I_{\sqcup B} = \bigcup_{x \in B} I_x$. We already know, that this situation takes place for continuous Scott domains.

Another class of domains, for which these properties can be established, is the class of coherently continuous dcpo's with countable basis. The term "coherence" here is understood in the weak sense of [1] (weaker, than bounded completeness), and not in the strong sense of [26] (stronger, than bounded completeness).

Definition 3.1. We say that a continuous dcpo A with the basis K is *coherently continuous*, if for any two basic elements $k, k' \in K$, the set of their minimal upper bounds, $\mathrm{MUB}(k, k')$, is finite, and for any $x \in A$, if $k \sqsubseteq x$ and $k' \sqsubseteq x$, then there is $k'' \in \mathrm{MUB}(k, k')$, such that $k'' \sqsubseteq x$.

Theorem 12.3.1 *If A is a coherently continuous dcpo with countable basis K, then for any $x \in A$, I_x is open.*

Proof. Consider $y \in I_x$. Since the space has a countable basis, we only need to show, that if there is a sequence of basic elements, $k_1 \sqsubseteq k_2 \sqsubseteq \ldots$, such that $y = \sqcup k_i$, then some k_i belongs to I_x.

By contradiction, assume that this is not the case. Then for all i, k_i and x have an upper bound. Using the presence of a countable basis again, approximate x with a sequence of basic elements as well: $l_1 \sqsubseteq l_2 \sqsubseteq \ldots$, $x = \sqcup l_i$. Then for all i, k_i and l_i have an upper bound.

Now we are going to build the sequence of (not necessarily basic) elements, $u_1 \sqsubseteq u_2 \sqsubseteq \ldots$, such that for all i, $k_i \sqsubseteq u_i$ and $l_i \sqsubseteq u_i$. Then $\sqcup u_i$ would be an upper bound of x and y, yielding the desired contradiction.

Consider an element $v \in \mathrm{MUB}(k_i, l_i)$. Define the height of v as maximal j, such that there is $w \in \mathrm{MUB}(k_j, l_j)$, such that $v \sqsubseteq w$. If there is no such maximal natural number, we say that v is of infinite height. Using coherence condition, it is easy to see, that there is an element $u_1 \in \mathrm{MUB}(k_1, l_1)$ of infinite height. Now consider only elements

$v \in \mathrm{MUB}(k_2, l_2)$, such that $u_1 \sqsubseteq v$. Using coherence condition, it is easy to see once again, that there is an element $u_2 \in \mathrm{MUB}(k_2, l_2)$ of infinite height, such that $u_1 \sqsubseteq u_2$. Continuing this process, we obtain the desired sequence. $\square$

Therefore, one can use $N_x = I_x$ in order to obtain all results of the previous section not only for continuous Scott domains, but also for coherently continuous dcpo's with countable bases.

12.4 When I_x Behaves Badly

12.4.1 Example

Let start with the example. We define an algebraic countable dcpo E, as a following subset of the powerset of $\mathbf{Z}$, ordered by subset inclusion. $E = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \ldots,$ $\{1,2,3,\ldots\}, \{0\}, \{1,0,-1\}, \{1,2,0,-2\}, \{1,2,3,0,-3\}, \ldots\}$. For convenience, we introduce a unique letter denotation for each of the elements of E: $\perp_E = \emptyset$, $e_1 = \{1\}$, $e_2 = \{1,2\}$, $e_3 = \{1,2,3\}, \ldots, e_\infty = \{1,2,3,\ldots\}$, $0_E = \{0\}$, $f_1 = \{1,0,-1\}$, $f_2 = \{1,2,0,-2\}$, $f_3 = \{1,2,3,0,-3\}, \ldots$ We will use this notation throughout the chapter. Observe, that all elements, except e_∞, are compact, that elements $e_\infty, f_1, f_2, \ldots$ are total, and that $e_i \sqsubseteq f_j$ iff $i \leq j$.

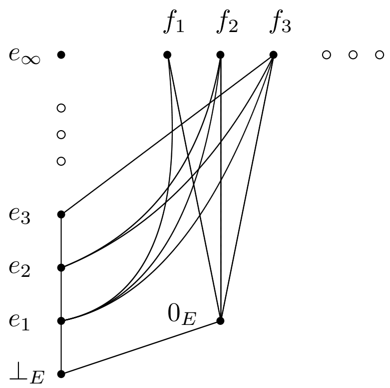


Figure 12.1: Domain E

Now we will see how *negation duality* works in this example. In our previous terminology, $x = e_\infty$, $y = 0_E$. The role of a directed set B is played by an increasing sequence, $e_1 \sqsubseteq e_2 \sqsubseteq e_3 \sqsubseteq \ldots$. Notice that $e_\infty = \sqcup B$.

Note also that $I_{0_E} = \{e_\infty\}$ and $0_E \in I_{e_\infty}$. You see, that I_{0_E} is not Scott open (although I_{e_∞} is Scott open), and dually, we obtain that $0_E \notin \bigcup_{b \in B} I_b$ (due to observation, that $I_{e_1} = \emptyset, I_{e_2} = \{f_1\}, I_{e_3} = \{f_1, f_2\}, \ldots$), thus breaking $I_{\sqcup B} = \bigcup_{b \in B} I_b$.

The breaking of this equality prevents the resulting relaxed metrics from being Scott continuous, as 0_E is compact and should naturally carry a finite weight. Since all I_b and $I_{\sqcup B}$ are Scott open, taking $\mathrm{Int}(I_x)$ as N_x instead of I_x would not fix this problem.

12.4.2 Solution

Let us rewrite the negation duality as $I_x = \{y \mid x \in I_y\}$. What works, somewhat surprisingly, is taking a subset of I_x via taking the interior inside the right-hand side of this expression: $J_x = \{y \mid x \in \mathrm{Int}(I_y)\}$.

Lemma 12.4.1 *If B is a directed set, $J_{\sqcup B} = \bigcup_{b \in B} J_b$.*

Proof. A potentially non-trivial part is to prove $J_{\sqcup B} \subseteq \bigcup_{b \in B} J_b$. Consider $y \in J_{\sqcup B}$. By definition of J, $\sqcup B \in \mathrm{Int}(I_y)$. Since $\mathrm{Int}(I_y)$ is Scott open, there is $b \in B$, such that $b \in \mathrm{Int}(I_y)$, that is $y \in J_b$. $\square$

In the next subsection, we explain this result in terms of Stone duality. In our example domain E, J_{e_∞} does not include 0_E, unlike I_{e_∞}, yielding $J_{e_\infty} = \bigcup_i J_{e_i}$.

In general, J_x is Alexandrov open, but does not have to be Scott open. E.g., in our example domain E, we have that $J_{0_E} = I_{0_E} = \{e_\infty\}$, and thus J_{0_E} is not Scott open. Due to co-continuity we can extend μ to Alexandrov open sets V, by defining $\mu(V) = \mu(\mathrm{Int}(V))$.

However, in order to use a setup of Section 11.1.4, it is much more convenient to define $N_x = \mathrm{Int}(J_x)$ and to use the following Lemma.

Lemma 12.4.2 *If for arbitrary Alexandrov open sets $J, J_m, m \in M$, the equality $J = \bigcup_{m \in M} J_m$ holds, then $\mathrm{Int}(J) = \bigcup_{m \in M} \mathrm{Int}(J_m)$ holds as well.*

Proof. The potentially non-trivial direction is to prove $\mathrm{Int}(J) \subseteq \bigcup_{m \in M} \mathrm{Int}(J_m)$. Consider $y \in \mathrm{Int}(J)$. By the Border Lemma (Lemma 9.2.1) there is $x \in J$, such that $x \ll y$. Then, because of the condition of the Lemma we are currently proving, there is $m \in M$, such that $x \in J_m$. Then applying the Border Lemma again, we obtain $y \in \mathrm{Int}(J_m)$. □

Hence, $N_x = \mathrm{Int}(J_x)$ enables us to satisfy all the requirements of the setup of Section 11.1.4, except for the requirement that for all $x \in \mathit{Total}(A), C_x \cup N_x = A$, which does not hold in general. E.g. consider our example domain E, and observe that $e_\infty \in \mathit{Total}(A)$, but $0_E \notin C_x \cup N_x$. (Observe, also that changing N_x to J_x does not fix this.)

Thus the Theorem 11.1.1 holds, but the Theorem 11.1.2 about the equality of $l(x, y)$ and $u(x, y)$ does not have to hold, and the resulting induced metric on $\mathit{Total}(A)$ cannot in general be obtained. E.g., in our example domain E, we have that $e_\infty \in \mathit{Total}(E)$, but $u(e_\infty, e_\infty) = \mu(I_{e_\infty}) - \mu(J_{e_\infty}) = \mu(\{0_E\})$, which is, in general, not zero, since 0_E is compact.

12.4.3 Stone Duality

The first Lemma in the previous subsection holds for the reasons, which are not related to such specific features of J_x as the use of $\mathrm{Int}(I_y)$ (any open set can be used instead) and the fact, that x and y belong to the same set A.

We analyze this situation in the spirit of *Stone duality* [30, 55], which is a contravariant equivalence between categories of spatial frames (of open sets) and sober topological spaces.

For the purpose of this subsection only, assume that there is a continuous dcpo A (Scott topologies of continuous dcpo's are sober [30]) and a set D, and that we are given a map $U : D \to \mathcal{O}(A)$, where $\mathcal{O}(A)$ is the frame of Scott open sets of the domain A.

Now generalize the construction of J_x by considering the map $J : A \to \mathcal{P}(D)$, where $\mathcal{P}(D)$ is a powerset of D ordered by set-theoretic inclusion and equipped with the Scott topology. Define J by the formula: $J : x \mapsto \{y \in D \,|\, x \in U(y)\}$. Then observe that the proof of Lemma 12.4.1 still goes through, implying that J is a Scott continuous function from A to $\mathcal{P}(D)$.

Now observe, if one applies J^{-1} to a subbasic open set $\uparrow\{y\}$, one obtains $J^{-1}(\Uparrow\{y\}) = \{x \mid J(x) \in \Uparrow\{y\}\} = \{x \mid y \in J(x)\} = \{x \mid x \in U(y)\} = U(y)$.

Thus the map U can be thought of as defined on the generators $\uparrow\{y\}$ of the free frame of all Scott open sets on $\mathcal{P}(D)$ and giving raise to the frame homomorphism $u : \mathcal{O}(\mathcal{P}(D)) \to \mathcal{O}(A)$ (of course, $u = J^{-1}$, e.g. for basic open sets, $u(\Uparrow\{y_1, \ldots, y_n\}) = U(y_1) \cap \ldots \cap U(y_n)$, and the similar thing goes for unions of basic sets).

Now, since we are dealing with sober spaces, Stone duality means, that not only $u = J^{-1}$ can be obtained from the continuous function J, but also the continuous function J can be restored from the frame morphism u. And this is the essence of our definition of J, when we think about U as defined on the generators of the frame $\mathcal{O}(\mathcal{P}(D))$.

12.4.4 J_x Is the Largest Continuous Approximation of I_x

Both I_x and J_x can be considered as functions from A to the powerset of A, $\mathcal{P}(A)$. However, in general, only J_x is Scott continuous. The following theorem shows that, in some sense, J_x is the best we can do.

Theorem 12.4.1 *If $f : A \to \mathcal{P}(A)$ is a Scott continuous function and $\forall x \in A.\, f(x) \subseteq I_x$, then $\forall x \in A.\, f(x) \subseteq J_x$.*

Proof. Assume that such Scott continuous function f is given, and for some x, there is $y \in f(x)$, such that $y \notin J_x$, i.e. $x \notin \mathrm{Int}(I_y)$. However, $y \in I_x$ means $x \in I_y$. Now consider a directed set $B = K_x$. We have that $x = \sqcup B$ and that all $b \in B$ are way below x. Then, taking into account $x \notin \mathrm{Int}(I_y)$ and applying the Border Lemma, we obtain that $\forall b \in B.\, b \notin I_y$.

However, the assumption of continuity of f means, that $f(\sqcup B) = \bigcup_{b \in B} f(b)$. Hence, since $y \in f(\sqcup B)$, there is some $b \in B$, such that $y \in f(b)$, hence $y \in I_b$, hence $b \in I_y$, contradicting the last formula of the previous paragraph. $\Box$

A similar statement holds for $\mathrm{Int}(I_x)$ and $\mathrm{Int}(J_x)$, understood as functions from A to the dual domain of open sets of A.

12.5 The Use of the Lawson Condition

We start with the equivalent formulation of the Lawson condition.

Lemma 12.5.1 *Given a continuous dcpo A, the relative Scott and Lawson topologies on $Total(A)$ coincide if and only if for all $x \in A$, $I_x \cap Total(A) = \mathrm{Int}(I_x) \cap Total(A)$.*

For example, in the domain E from the previous section Lawson condition does not hold. Indeed, $I_{0_E} = \{e_\infty\}$ and $e_\infty \in Total(E)$. However, $\mathrm{Int}(I_{0_E}) = \emptyset$. The set $\{e_\infty\}$ is open in the relative Lawson topology, but not in the relative Scott topology.

Now we are going to modify the domain E, in order to obtain a different example, which would satisfy the Lawson condition. We add a new element, $\star$, to $\mathbf{Z}$, so that E' will be a subset of the powerset of $\mathbf{Z} \cup \{\star\}$, ordered by the set-theoretic inclusion. Let $E' = E \cup \{e_\star\}$, where $e_\star = \{\star, 1, 2, 3, \ldots\}$.

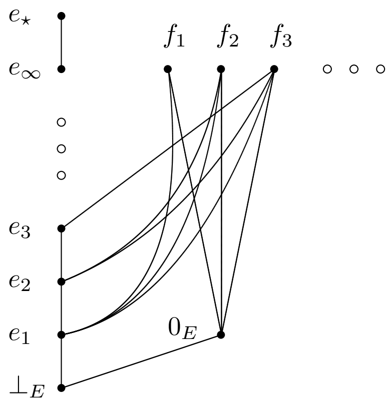


Figure 12.2: Domain E'

Now $I_{0_E} = \{e_\infty, e_\star\}$, so this is still not a Scott open set, however, $\mathrm{Int}(I_{0_E}) = \{e_\star\} = I_{0_E} \cap Total(E') = \mathrm{Int}(I_{0_E}) \cap Total(E')$. Notice, that $J_{0_E} = I_{0_E}$ here, so J_{0_E}

is still only Alexandrov open. J_{e_∞} is the same as in E, but now e_∞ is not a total element. However, $J_{e_\star} = J_{e_\infty} \cup \{0_E\}$, because $e_\star \in \mathrm{Int}(I_{0_E})$, so $J_{e_\star} = I_{e_\star} = \mathrm{Int}(J_{e_\star})$, and $C_{e_\star} \cup J_{e_\star} = C_{e_\star} \cup \mathrm{Int}(J_{e_\star}) = E'$. The set $\{e_\star\}$ is open in both Lawson and Scott relative topologies on $\mathit{Total}(E')$.

What is going on here is described by the following Theorem.

Theorem 12.5.1 *A continuous dcpo A satisfies the Lawson condition if and only if for all $x \in \mathit{Total}(A)$, $J_x = I_x$. Hence, if the Lawson condition holds, then $C_x \cup J_x = C_x \cup \mathrm{Int}(J_x) = A$.*

Proof. Assume that the Lawson condition holds and $x \in \mathit{Total}(A)$. Assume, that $y \not\sqsubseteq x$, i.e. $y \notin C_x$ and $y \in I_x$, using the totality of x. Thus, by duality, $x \in I_y$. Because $x \in \mathit{Total}(A)$ and because due to the Lawson condition $I_y \cap \mathit{Total}(A) = \mathrm{Int}(I_y) \cap \mathit{Total}(A)$, we obtain $x \in \mathrm{Int}(I_y)$, hence $y \in J_x$.

Conversely, assume $\forall x \in \mathit{Total}(A).\, J_x = I_x$. Let us prove $I_y \cap \mathit{Total}(A) = \mathrm{Int}(I_y) \cap \mathit{Total}(A)$, thus proving the Lawson condition. Take $x \in I_y \cap \mathit{Total}(A)$. By negation duality, $y \in I_x$, then, by totality of x and our assumptions, $y \in J_x$, which, by definition of J_x, means that $x \in \mathrm{Int}(I_y)$.

The rest follows from the observation, that if $x \in \mathit{Total}(A)$, I_x is Scott open. $\square$

Hence if the Lawson condition holds, the resulting μ*Info*-structure is not deficient, and the Theorem 11.1.2 holds.

12.6 Tolerances and Negative Information

12.6.1 Tolerances, Distinguishability, and Observability

Smyth [52] requires that a tolerance relation is closed in the product topology. Here are informal reasons for this.

The typical meaning of two points being in the relation of tolerance, $x \sim y$, is that x cannot be distinguished from y, i.e. there is no way to establish, that x and y differ.

The natural way to interpret the statement, that x and y can be distinguished, is to give some "effective" procedure for making such a distinction. Thus, the property of being distinguishable is observable [51]. Correspondingly, the property $x \sim y$ is refutable, hence $\sim$ should be closed.

The fact that the least closed tolerance for a weakly Hausdorff continuous dcpo is $\uparrow$ also is quite natural in this framework. Indeed, domain elements are thought of as being only partially known and dynamically increasing in the course of their lives. Hence the fact that $x \uparrow y$, that is $\exists z.\, x \sqsubseteq z, y \sqsubseteq z$, precisely means that x and y may approximate the same "genuine" element z, hence we cannot distinguish between them. Since $\uparrow$ is closed in the weakly Hausdorff case, its complement is open, hence observable. That means that when $x \uparrow y$ does not hold, there is some "finite" way to distinguish between x and y.

12.6.2 J' and the Least Closed Tolerance (a Smyth Conjecture)

Consider a continuous dcpo A. In this subsection $x, y, v, w \in A$. Recall that we defined $x \not\sim y = \{\langle x, y\rangle \mid x \in J'_y\}$.

Lemma 12.6.1 $x \not\sim y = \{\langle x, y\rangle \mid \exists \langle v, w\rangle.\, v \ll x, w \ll y, \{v, w\}$ *is unbounded*$\}$.

Proof. Using the Border Lemma we get $x \in \mathrm{Int}(J_y)$ iff $\exists v \in J_y.\, v \ll x$. By the definition of J_y, $v \in J_y$ iff $y \in \mathrm{Int}(I_v)$ i.e. $\exists w \in I_v.\, w \ll y$. Finally recall that $w \in I_v$ iff $\{v, w\}$ is unbounded . $\Box$

It is an immediate corollary that $\not\sim$ is symmetric.

Let us show that $\not\sim$ is open in the product topology. If we fix a pair $\langle v, w\rangle$ the set $\not\sim_{\langle v,w\rangle} = \{x \mid v \ll x\} \times \{y \mid w \ll y\}$ is open, and our $\not\sim$ is the union of all such sets for all unbounded pairs $\langle v, w\rangle$.

Theorem 12.6.1 *The relation $\not\sim$ is the complement of the least closed tolerance.*

Proof. Consider an open set $W \subseteq X \times X$, such that $\not\sim \subset W$. Consider a pair $\langle x, y\rangle \in W \setminus \not\sim$. Since W is open, we can choose two open sets $U \subseteq X$ and $V \subseteq X$, such that $\langle x, y\rangle \in U \times V \subseteq W$. Consider a pair $\langle p, r\rangle, p \in U, r \in V. p \ll x, r \ll y$. The pair $\langle p, r\rangle$ is bounded, otherwise $\langle x, y\rangle \in \not\sim$, so we can take z such that $p \sqsubseteq z$, $r \sqsubseteq z$, therefore $z \in U$, $z \in V$, so $\langle z, z\rangle \in U \times V$ and $\langle z, z\rangle \in W$. So the complement of W is not a tolerance, because it is not reflexive. $\square$

12.6.3 Examples

In our example domain E, the pair $\langle e_\infty, 0_E\rangle$ is unbounded, but belongs to the least closed tolerance, since these elements cannot be distinguished by looking at the approximation pairs, $\langle e_1, 0_E\rangle$, $\langle e_2, 0_E\rangle, \ldots$.

Moreover, by adding to domain E elements $g_2 = \{0, 2\}, g_3 = \{0, 2, 3\}, \ldots, g_n = \{0, 2, 3, \ldots, n\}$, ... and $g_\infty = \{0, 2, 3, 4, \ldots\}$, we obtain a domain, where two different maximal elements, e_∞ and g_∞, cannot be distinguished via finite observations, because all their approximations are bounded. Such a situation, when $\exists x, y \in \mathit{Total}(A).\, x \sim y$, where $\sim$ is the least closed tolerance, cannot occur in the presence of the Lawson condition, because the Lawson condition is equivalent to $\forall x \in \mathit{Total}(A).\, I_x = J'_x$.

12.7 Tolerances and Lower Bounds of Relaxed Metrics

We are going to prove the following statement.

Theorem 12.7.1 $x \not\sim y \Leftrightarrow l(x, y) \neq 0$.

Proof. $\Rightarrow$. Recall that $l(x, y) = \mu(C_x \cap J'_y) + \mu(C_y \cap J'_x)$. Notice that $x \not\sim y$ implies $x \in C_x \cap J'_y$, which implies that $C_x \cap J'_y \neq \emptyset$. Hence $J'_y \setminus C_x \subset J'_y$, hence $\mu(C_x \cap J'_y) = \mu(J'_y) - \mu(J'_y \setminus C_x) > 0$ due to the strong non-degeneracy of μ. Hence $l(x, y) > 0$.

$\Leftarrow$. $l(x, y) > 0$ means $C_x \cap J'_y \neq \emptyset$ or $C_y \cap J'_x \neq \emptyset$. It is enough to consider $C_x \cap J'_y \neq \emptyset$. Since x is the largest element of C_x, we obtain $x \in J'_y$, hence $x \not\sim y$. $\square$

Only upper bounds of relaxed metrics participate in the definition of the relaxed metric topology. Hence lower bounds are usually considered as only playing an auxiliary role in the computation of upper bounds. Here we see an example of a quite different application of lower bounds.

12.7.1 A Continuous Family of Tolerances

A set $\{\langle x, y\rangle \,|\, l(x, y) \leq \epsilon\}$, also forms a tolerance. Indeed, this is a symmetric, reflexive relation. To see that it is closed, consider a Scott continuous function $l : A \times A \to R^+$, where R^+ is a segment $[0, 1]$ with the usual ordering and the induced Scott topology, and observe that the set in question is the inverse image of a Scott closed set $[0, \epsilon] \subseteq R^+$ under l^{-1}.

The resulting family of tolerances parametrized by ϵ is Scott continuous in the following sense. The dual domain for R^+ is domain $R^-_\top$. Here R^- is the same segment of numbers, but with inverse ordering ($x \sqsubseteq y \Leftrightarrow x \geq y$), an element $r \in R^-$ corresponds to the open set $(r, 1] \subseteq R^+$, the element $\top \in R^-_\top$ corresponds to the open set $[0, 1] = R^+$, and domains R^- and $R^-_\top$ are equipped with the induced Scott topology.

The function $l^{-1} : R^-_\top \to \mathcal{O}(A \times A)$ is Scott continuous, and so is its restriction on R^-. Then $l^{-1}(\epsilon)$ is the complement of the tolerance in question, and we can think about this complement as representing this tolerance in the dual domain $\mathcal{O}(A \times A)$.

12.8 Open Problems

It might be useful to extend the Stone duality analysis to $\mathrm{Int}(J_x)$ and to be able to speak about the intuition behind the Lawson condition in the spirit of [51].

Another open question is as follows. If Lawson condition does not hold, can we obtain some negative results about the existence of $\mu Info$-structures with totality property, or, more generally, about the existence of relaxed metrics with the property $\forall x, y \in Total(A).\, l(x, y) = u(x, y)$? Obviously, this question allows a number of variations, e.g. we know now, that when this question is restricted to the case of $Info(x) = C_x$, such negative results can indeed be obtained.

It seems that tolerances will play an increasingly important role in domain theory. One particularly promising direction of development is to use tolerances and especially their asymmetric generalizations instead of transitivity of logical inference to formally express and study the ideas of A.S.Esenin-Vol'pin and P.Vopenka, that large numbers should be considered infinite, and long proofs and computations should be considered meaningless [56].

Part IV

Conclusion

In the preface we identified the key unsolved **Problem A**, namely to learn how to find reasonable approximations for a sufficiently wide class of definitions of domain elements and Scott continuous functions while spending realistic amount of resources, as the main obstacle on the path of wider practical applicability of domains with Scott topology in software engineering.

In our work we achieved considerable progress in the development of analysis on such domains. In conclusion, we would like to stress the extreme importance of finding analogs of various series decompositions in domains with Scott topologies. The theory of such decompositions should incorporate the Scott domain analogs of such seemingly unrelated things as Fourier series and decimal representations of real numbers and would most likely represent a considerable step towards solving the Problem A.

This direction of research is also closely related to another aspect of Problem A, namely how to find compact visual representations of domain elements. Such representations, of course, should denote relatively close approximations of the elements in question.

Failing the solution of Problem A, it is possible that certain mathematical constructions in domain theory might give us some hints on how to construct novel algorithms.

Part V

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