Propositional Logic

CS 112
Fall 2006

Types of Knowledge

- Procedural, e.g.: functions
  Such knowledge can only be used in one way -- by executing it

- Declarative, e.g.: constraints
  It can be used to perform many different sorts of inferences

Logic

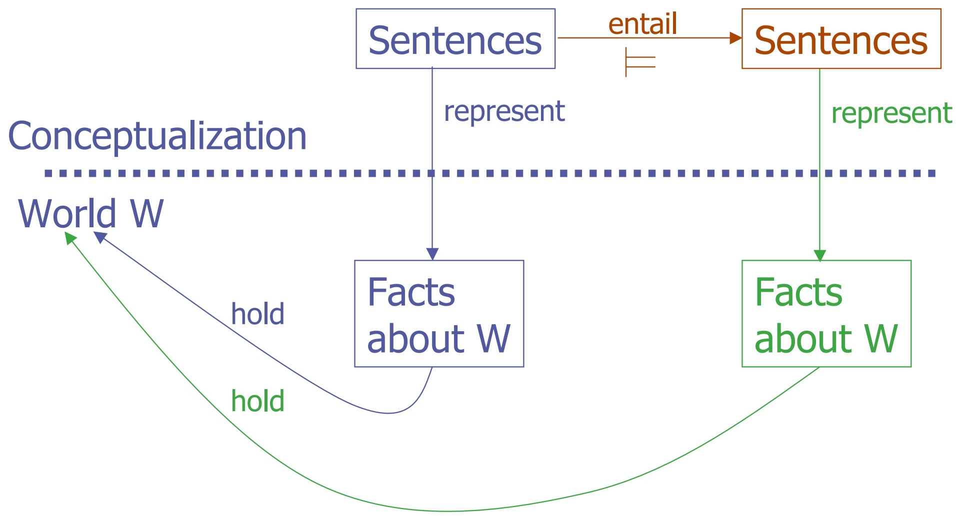
Logic is a declarative language to:

- Assert sentences representing facts that hold in a world W (these sentences are given the value true)
- Deduce the true/false values to sentences representing other aspects of W

Logic in general

- Logics are formal languages for representing information such that conclusions can be drawn
- Syntax defines the sentences in the language
- Semantics define the "meaning" of sentences;
  - i.e., define truth of a sentence in a world

Connection World-Representation



Examples of Logics

- Propositional calculus

  $A \wedge B \Rightarrow C$

- First-order predicate calculus

  $(\forall x)(\exists y)\ Mother(y,x)$

- Logic of Belief

  B(John,Father(Zeus,Cronus))

Model

- A model of a sentence is an assignment of a truth value – true or false – to every atomic sentence such that the sentence evaluates to true.

Model of a KB

- Let KB be a set of sentences

- A model m is a model of KB iff it is a model of all sentences in KB, that is, all sentences in KB are true in m.

Satisfiability of a KB

A KB is satisfiable iff it admits at least one model; otherwise it is unsatisfiable

KB1 = $\{P, \neg Q \wedge R\}$ is satisfiable

KB2 = $\{\neg P \vee P\}$ is satisfiable

KB3 = $\{P, \neg P\}$ is unsatisfiable

valid sentence or tautology

Logical Entailment

- KB : set of sentences
- α : arbitrary sentence
- KB entails α – written KB $\models \alpha$ – iff every model of KB is also a model of α
- Alternatively, KB $\models \alpha$ iff
  - $\{KB, \neg\alpha\}$ is unsatisfiable
  - KB $\Rightarrow \alpha$ is valid

Inference Rule

- An inference rule $\{\xi, \psi\} \vdash \varphi$ consists of 2 sentence patterns ξ and ψ called the conditions and one sentence pattern φ called the conclusion
- If ξ and ψ match two sentences of KB then the corresponding φ can be inferred according to the rule

Inference

- I: Set of inference rules
- KB: Set of sentences
- Inference is the process of applying successive inference rules from I to KB, each rule adding its conclusion to KB

Example: Modus Ponens

$$\{\ \alpha \Rightarrow \beta\ ,\ \alpha\ \}\ \vdash\ \beta$$

$$\{\xi,\ \psi\}\ \vdash \varphi$$

From

Battery-OK ∧ Bulbs-OK ⇒ Headlights-Work

Battery-OK ∧ Bulbs-OK

Infer

Headlights-Work

⇒ Connective symbol (implication)

⊨ Logical entailment

KB ⊨ α iff KB ⇒ α is valid

⊢ Inference

Soundness

- An inference rule is sound if it generates only entailed sentences
- All inference rules previously given are sound, e.g.:
  modus ponens: $\{\alpha \Rightarrow \beta , \alpha\} \vdash \beta$
- The following rule:
  $$\{\alpha \Rightarrow \beta , \beta\} \vdash \alpha$$
  is unsound, which does not mean it is useless (an inference rule for *abduction*, outside scope of this course)

Is each of the following a sound inference rule?

$$\{\alpha \Rightarrow \beta , \neg\alpha\} \vdash \neg\beta$$

$$\{\alpha \Rightarrow \beta , \neg\beta\} \vdash \neg\alpha$$

Completeness

- A set of inference rules is complete if every entailed sentences can be obtained by applying some finite succession of these rules
- Modus ponens *alone* is not complete, e.g.:
  from $A \Rightarrow B$ and $\neg B$, we cannot get $\neg A$

Proof

The proof of a sentence α from a set of sentences KB is the derivation of α by applying a series of sound inference rules

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1.	Battery-OK ∧ Bulbs-OK ⇒ Headlights-Work	
2.	Battery-OK ∧ Starter-OK ∧ ¬Empty-Gas-Tank ⇒ Engine-Starts	
3.	Engine-Starts ∧ ¬Flat-Tire ⇒ Car-OK	
4.	Headlights-Work	
5.	Battery-OK	
6.	Starter-OK	
7.	¬Empty-Gas-Tank	
8.	¬Car-OK	
9.	Battery-OK ∧ Starter-OK	by 5,6
10.	Battery-OK ∧ Starter-OK ∧ ¬Empty-Gas-Tank	by 9,7
11.	Engine-Starts	by 2,10
12.	Engine-Starts ⇒ Flat-Tire	by 3,8
13.	Flat-Tire	by 11,12

Inference Problem

- Given:
  - KB: a set of sentence
  - α: a sentence
- Answer:
  - KB $\models \alpha$?

Deduction vs. Satisfiability Test

KB $\models \alpha$ iff $\{KB, \neg\alpha\}$ is unsatisfiable

Hence:

- Deciding whether a set of sentences entails another sentence, or not
- Testing whether a set of sentences is satisfiable, or not

are closely related problems