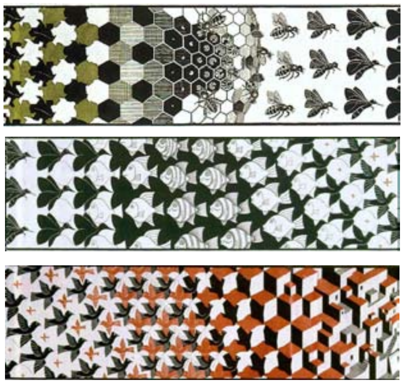
2D Geometrical Transformations

Foley & Van Dam, Chapter 5



2D Geometrical Transformations

- Translation
- Scaling
- Rotation
- Shear
- Matrix notation
- Compositions
- Homogeneous coordinates

2D Geometrical Transformations

Assumption: Objects consist of points and lines.
A point is represented by its Cartesian coordinates:

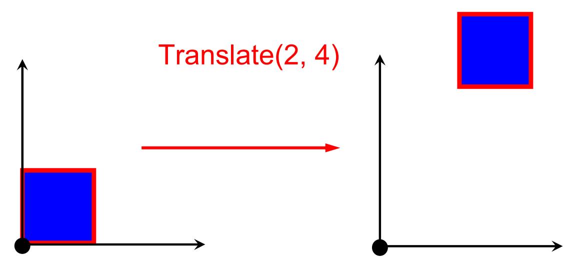
$$P = (x, y)$$

Geometrical Transformation:
Let (A, B) be a straight line segment between the points A and B.
Let T be a general 2D transformation.
T transforms (A, B) into another straight line segment (A', B'), where:

$$A' = TA \text{ and}$$
$$B' = TB$$

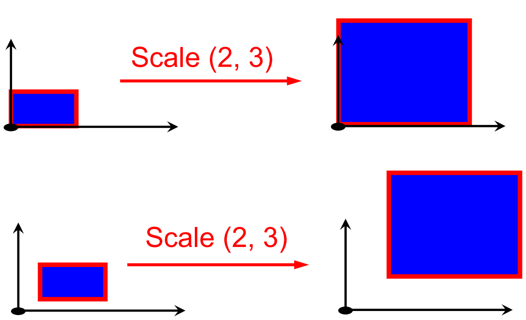
Translation

- Translate(a, b): $(x, y) \rightarrow (x+a, y+b)$



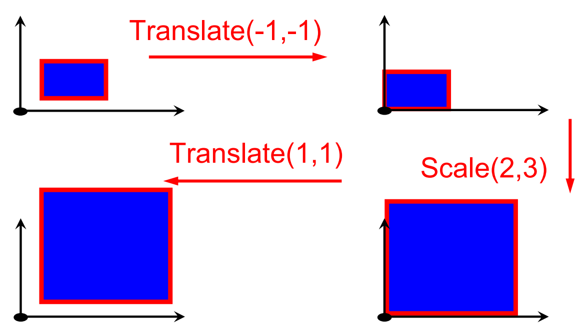
Scale

- Scale (a, b): $(x, y) \rightarrow (ax, by)$



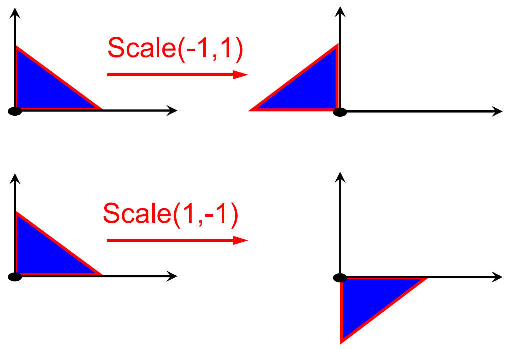
Scale

- How can we scale an object without moving its origin (lower left corner)?



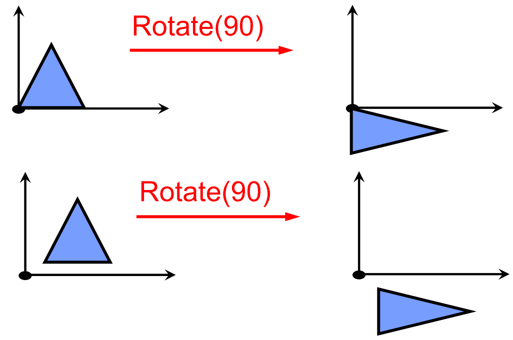
Reflection

- Special case of scale



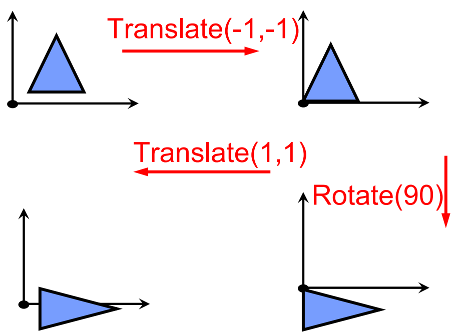
Rotation

- Rotate(θ):
  $(x, y) \rightarrow (x \cos(\theta)+y \sin(\theta), -x \sin(\theta)+y \cos(\theta))$



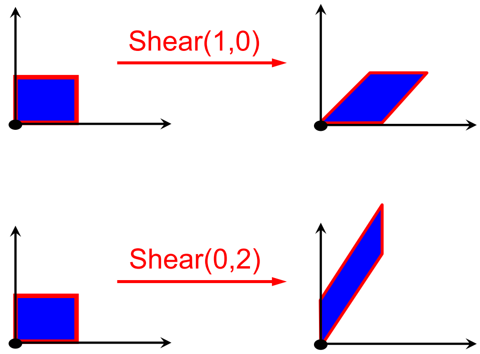
Rotation

- How can we rotate an object without moving its origin (lower left corner)?



Shear

- Shear (a, b): $(x, y) \rightarrow (x+ay, y+bx)$

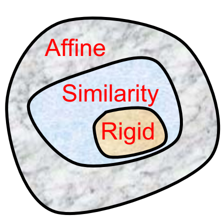


Classes of Transformations

- Rigid transformation (distance preserving):
  Translation + Rotation
- Similarity transformation (angle preserving):
  Translation + Rotation + Uniform Scale
  - Affine transformation (parallelism preserving):
    Translation + Rotation + Scale + Shear

All above transformations are groups where

Rigid $\subset$ Similarity $\subset$ Affine



Matrix Notation

- Let's treat a point (x, y) as a 2x1 matrix (column vector):

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

- What happens when this vector is multiplied by a 2x2 matrix?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

2D Transformations

- 2D object is represented by points and lines that join them
- Transformations can be applied only to the the points defining the lines
- A point (*x, y*) is represented by a 2x1 column vector, so we can represent 2D transformations by using 2x2 matrices:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale

- Scale (a, b): (x, y) → (ax, by)

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

- If *a* or *b* are negative, we get reflection
- Inverse: $S^{-1}(a,b) = S(1/a, 1/b)$

Reflection

- Reflection through the *y* axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Reflection through the *x* axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Reflection through *y* = *x*:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Reflection through *y* = -*x*:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Shear

- Shear (a, b): (x, y) → (x+ay, y+bx)

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y + bx \end{bmatrix}$$

Rotation

- Rotate(θ):

(x, y) → (x cos(θ)+y sin(θ), -x sin(θ)+y cos(θ))

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

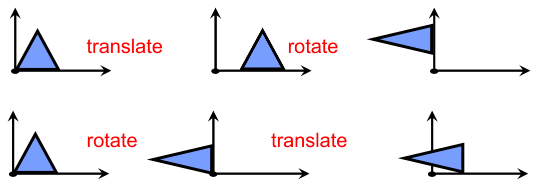
- Inverse: $R^{-1}(q) = R^{T}(q) = R(-q)$

Composition of Transformations

- A sequence of transformations can be collapsed into a single matrix:

$$[A][B][C]\begin{bmatrix} x \\ y \end{bmatrix} = [D]\begin{bmatrix} x \\ y \end{bmatrix}$$

- Note: Order of transformations is important!



Translation

- Translation (a, b):

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

Problem: Cannot represent translation using 2x2 matrices

Solution:

Homogeneous Coordinates

Homogeneous Coordinates

Is a mapping from R^n to R^{n+1}:

$$(x, y) \to (X, Y, W) = (tx, ty, t)$$

Note: All triples (tx, ty, t) correspond to the same non-homogeneous point (x, y)
Example (2, 3, 1) ≡ (6, 9, 3).

Inverse mapping:

$$(X, Y, W) \to \left(\frac{X}{W}, \frac{Y}{W}\right)$$

Translation

- Translate(a, b):

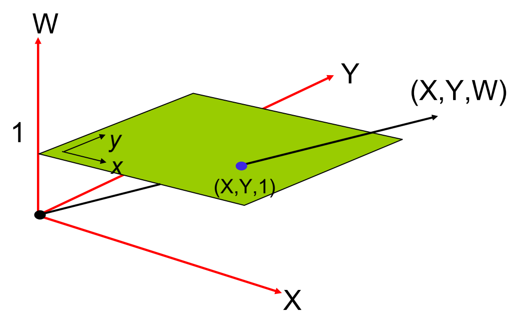
$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ 1 \end{bmatrix}$$

Inverse: $T^{-1}(a, b) = T(-a, -b)$
Affine transformations now have the following form:

$$\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix}$$

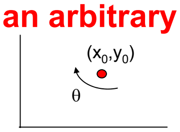
Geometric Interpretation

A 2D point is mapped to a line (ray) in 3D
The non-homogeneous points are obtained by projecting the rays onto the plane Z=1



Example

Rotation about an arbitrary point



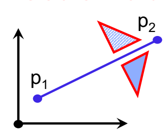
1. Translate the coordinates so that the origin is at (x_0, y_0)
2. Rotate by θ
3. Translate back

$$\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_0(1-\cos\theta) + y_0 \sin\theta \\ \sin\theta & \cos\theta & y_0(1-\cos\theta) - x_0 \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Example

Reflection about an arbitrary line



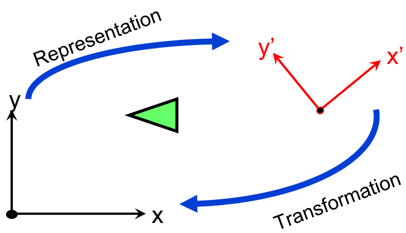
$L = p_1 + t(p_2 - p_1) = t p_2 + (1-t) p_1$

1. Translate the coordinates so that P_1 is at the origin
2. Rotate so that L aligns with the x-axis
3. Reflect about the x-axis
4. Rotate back
5. Translate back

Change of Coordinates

It is often required to transform the description of an object from one coordinate system to another

Rule: Transform one coordinate frame towards the other in the opposite direction of the representation change



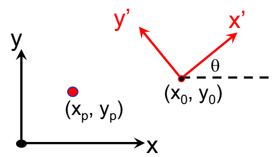
Example

• **Change of coordinates:** Represent P = (x_p, y_p, 1) in the (x', y') coordinate system

$$P' = MP$$

Where:

$$M = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



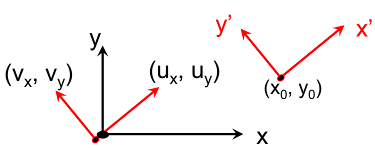
Example

• **Change of coordinates:**
Alternative method: assume x' = (ux, uy) and y' = (vx, vy) in the (x, y) coordinate system

where

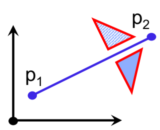
$$P' = MP$$

$$M = \begin{pmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



Example

Reflection about an arbitrary line



$L = p_1 + t\,(p_2 - p_1) = t\,p_2 + (1-t)\,p_1$

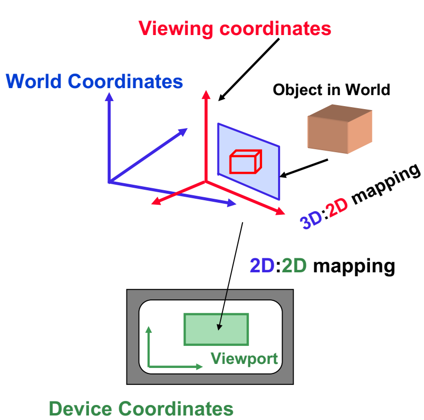
Define a coordinate systems (u, v) parallel to P_1P_2:

$$\mathbf{u} = \frac{p_2 - p_1}{|p_2 - p_1|} \equiv \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & p_{1_x} \\ 0 & 1 & p_{1_y} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x & v_x & 0 \\ u_y & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -p_{1_x} \\ 0 & 1 & -p_{1_y} \\ 0 & 0 & 1 \end{pmatrix}$$

3D Viewing Transformation Pipeline

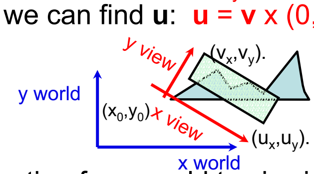


World to Viewing Coordinates

In order to define the viewing window we have to specify:

- •Windowing-coordinate **origin** $P_0 = (x_0, y_0)$
- •View **vector up** **v** $= (v_x, v_y)$

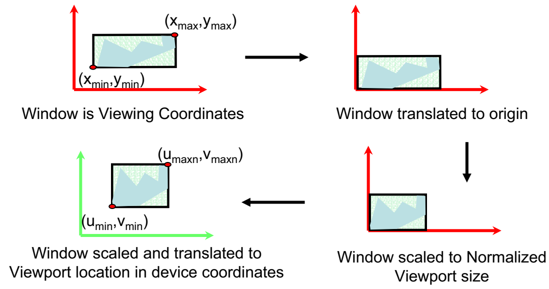
•Using **v**, we can find **u**: **u = v** x (0,0,1)



Transformation from world to viewing coordinates :

$$M_{wc-vc} = \begin{pmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

Window to Viewport Coordinates



$$M_{vc-dc} = \begin{pmatrix} 1 & 0 & u_{min} \\ 0 & 1 & v_{min} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{max} - u_{min} & 0 & 0 \\ 0 & v_{max} - v_{min} & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1}{x_{max} - x_{min}} & 0 & 0 \\ 0 & \frac{1}{y_{max} - y_{min}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_{min} \\ 0 & 1 & -y_{min} \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Normalized Device Coordinates}}$$

Efficiency Considerations

A 2D point transformation requires 9 multiplies and 6 adds

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

But since affine transformations have always the form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

The number of operations can be reduced to 4 multiplies and 4 adds

Efficiency Considerations

The rotation matrix is:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

When rotating of small angles θ, we can use the fact that *cos*(θ) $\cong$ 1 and simplify

$$\begin{bmatrix} 1 & \sin\theta \\ -\sin\theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y\sin\theta \\ -x\sin\theta + y \end{bmatrix}$$

Determinant of a Matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

If P is a polygon of area A_P, transformed by a matrix M, the area of the transformed polygon is $A_P * |M|$