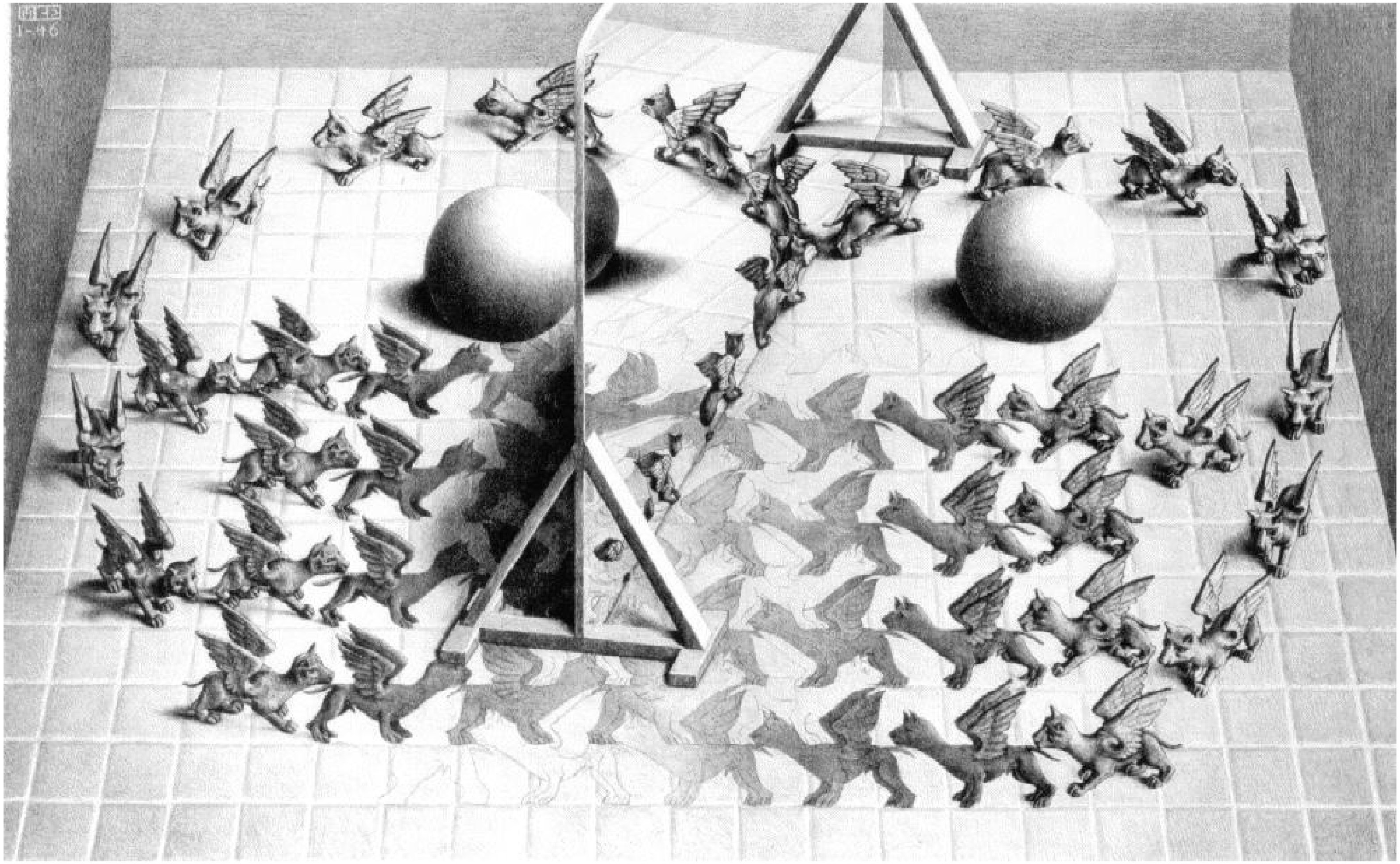
3D Geometrical Transformations

Foley & Van Dam, Chapter 5

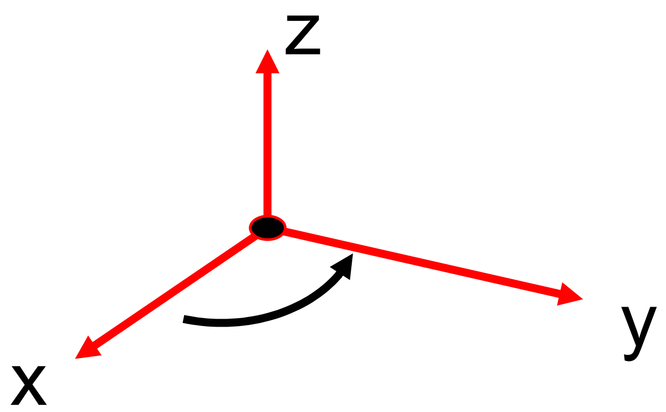


3D Geometrical Transformations

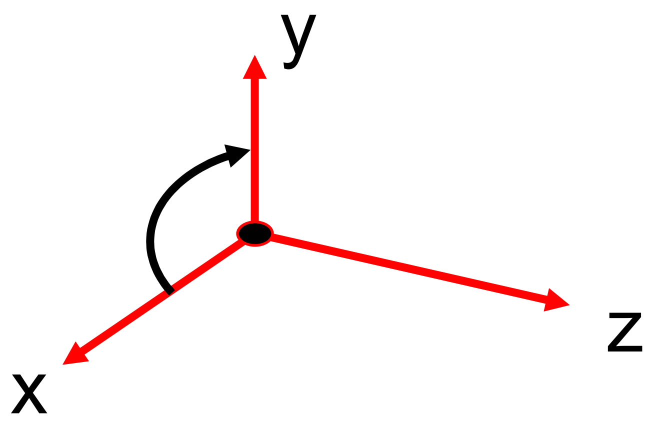
- 3D point representation
- Translation
- Scaling, reflection
- Shearing
- Rotations about x, y and z axis
- Composition of rotations
- Rotation about an arbitrary axis
- Transforming planes

3D Coordinate Systems

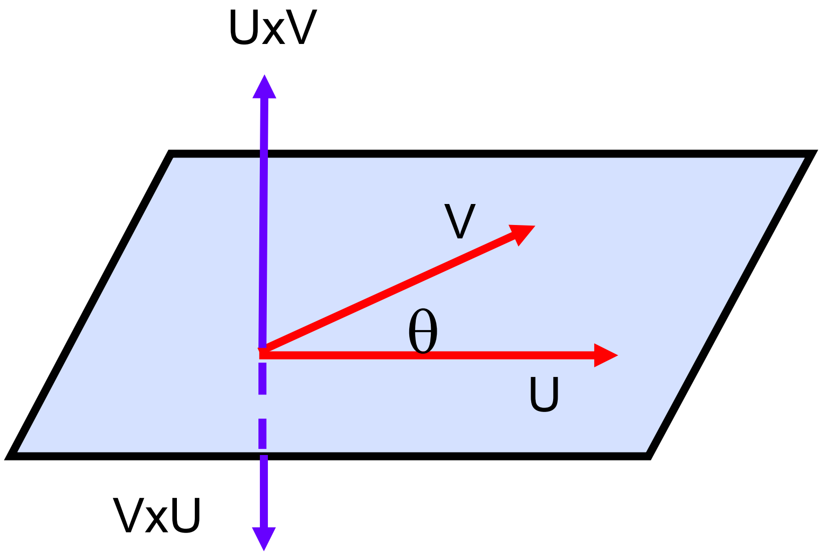
Right-handed coordinate system:



Left-handed coordinate system:



Reminder: Cross Product



$$U \times V = \hat{n} \, |U| \, |V| \sin\theta$$

$$U \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

3D Point Representation

- A 3D point *P* is represented in homogeneous coordinates by a 4-dimensional vector:

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- As for 2D points:

$$p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \equiv \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \\ \alpha \end{bmatrix}$$

3D Geometrical Transformations

- In homogeneous coordinates, 3D affine transformations are represented by 4x4 matrices:

$$\begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A point transformation is performed:

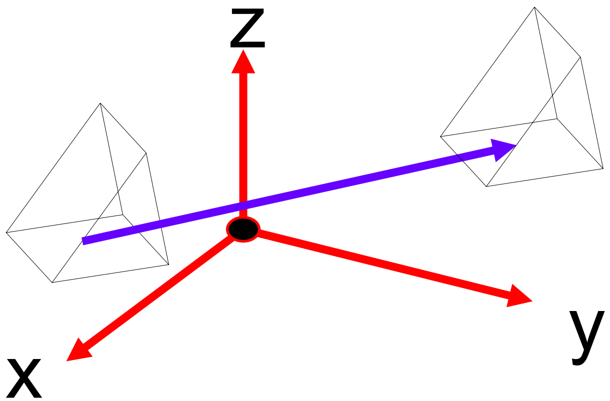
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Translation

P in translated to P' by:

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

Or: $T \; P = P'$

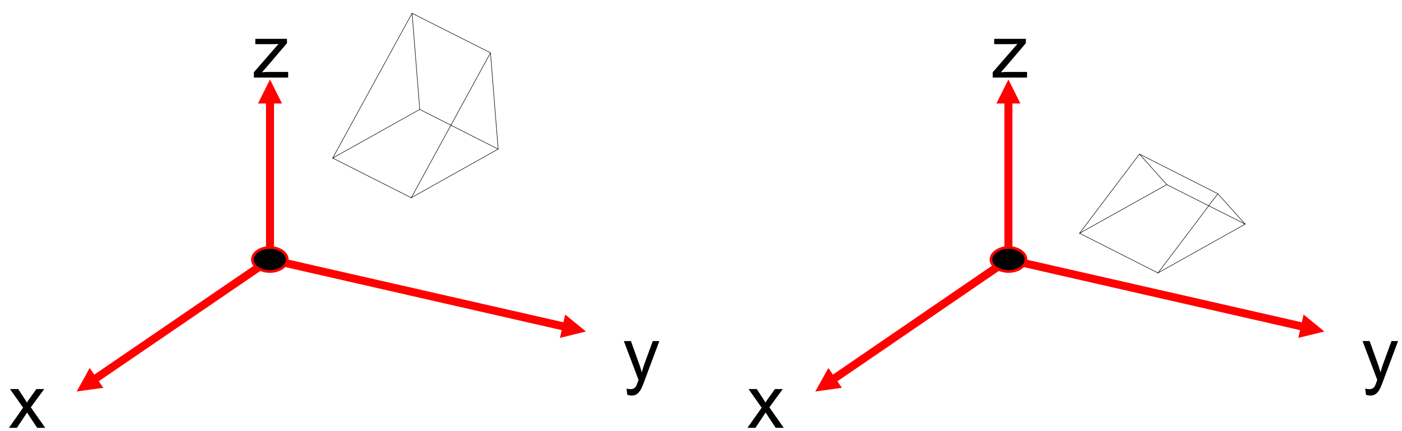


Inverse: $T^{-1} P' = P$

3D Scaling

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$$

Or $$S\ P = P'$$



$$S^{-1} P' = P$$

3D Reflection

- A reflection through the *xy* plane:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z \\ 1 \end{bmatrix}$$

- Reflections through the xz and the yz planes are defined similarly

3D Shearing

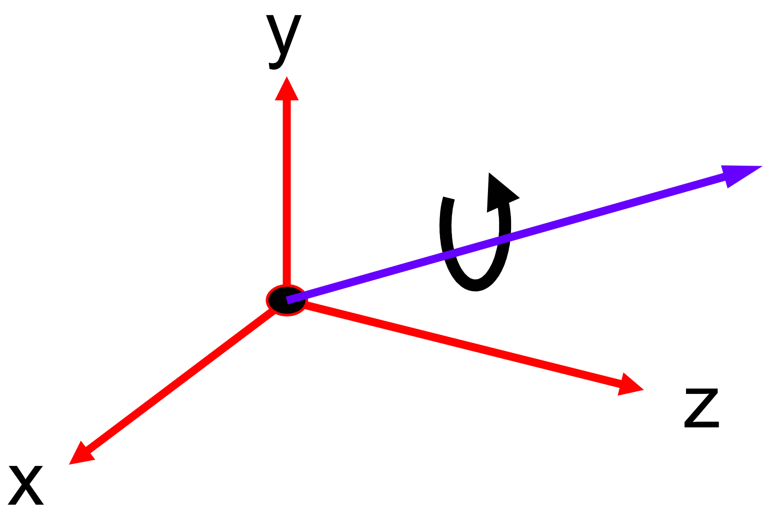
- Shearing:

$$\begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + ay + bz \\ cx + y + dz \\ ex + fy + z \\ 1 \end{bmatrix}$$

- Change in each coordinate is a linear combination of all three
- Transforms a cube into a general parallelepiped

3D Rotation

- To generate a rotation in 3D we have to specify:
  - axis of rotation (2 d.o.f.)
  - amount of rotation (1 d.o.f.)
- Note, the axis passes through the origin

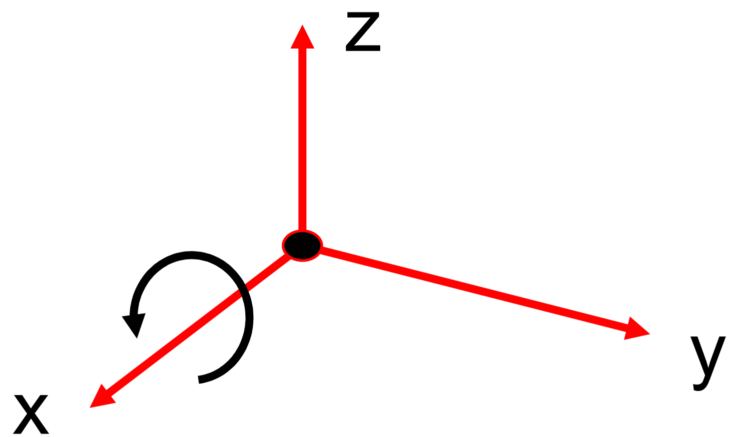


3D Rotation

- Counterclockwise rotation about x-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$p' = R_x(\theta)\,p$$

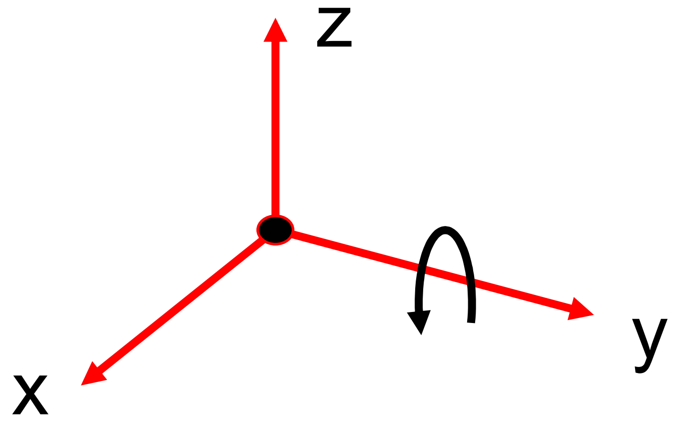


3D Rotation

- Counterclockwise rotation about y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$p' = R_y(\theta)\, p$$

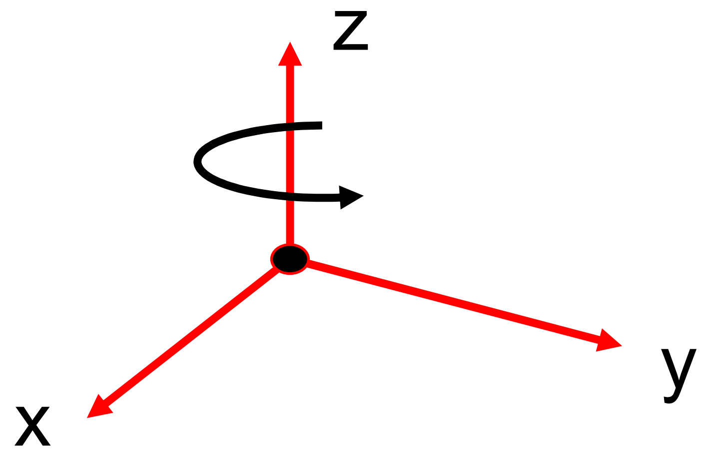


3D Rotation

- Counterclockwise rotation about z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$p' = R_z(\theta)\, p$$



Composite Rotation

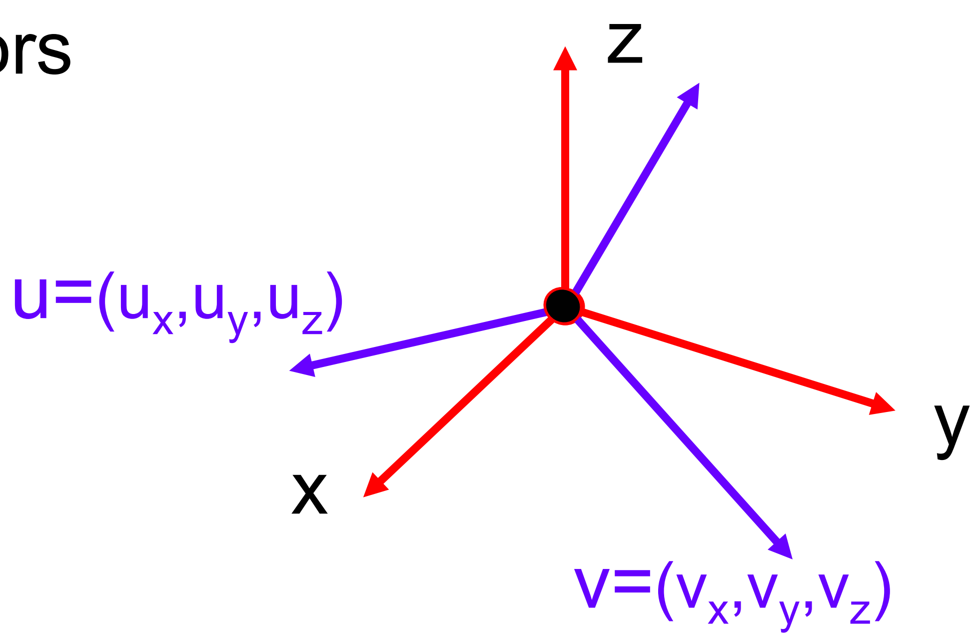
- Rx, Ry, and Rz, can perform *any* rotation about an axis passing through the origin

- Inverse rotation:

$$p = R^{-1}(\theta)\, p' = R(-\theta)\, p' = R^{T}(\theta)$$

Change of Coordinates

- **Problem:** Given the *XYZ* orthonormal coordinate system, find a transformation M, that maps a representation in *XYZ* into a representation in the orthonormal system *UVW*, with the same origin
- The matrix *M* transforms the *UVW* vectors to the *XYZ* vectors



Change of Coordinates

- **Solution:** M is rotation matrix whose rows are *U*,*V*, and *W*:

$$M = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Note: the inverse transformation is the transpose:

$$\mathrm{M}^{-1} = \mathrm{M}^{\mathrm{T}} = \begin{bmatrix} \mathrm{u}_x & \mathrm{v}_x & \mathrm{w}_x & 0 \\ \mathrm{u}_y & \mathrm{v}_y & \mathrm{w}_y & 0 \\ \mathrm{u}_z & \mathrm{v}_z & \mathrm{w}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Change of Coordinates

- Let's check the transformation of *U* under *M*:

$$MU = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_x^2 + u_y^2 + u_z^2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = X$$

- Similarly, V goes into Y, and W goes into Z

Change of Coordinates

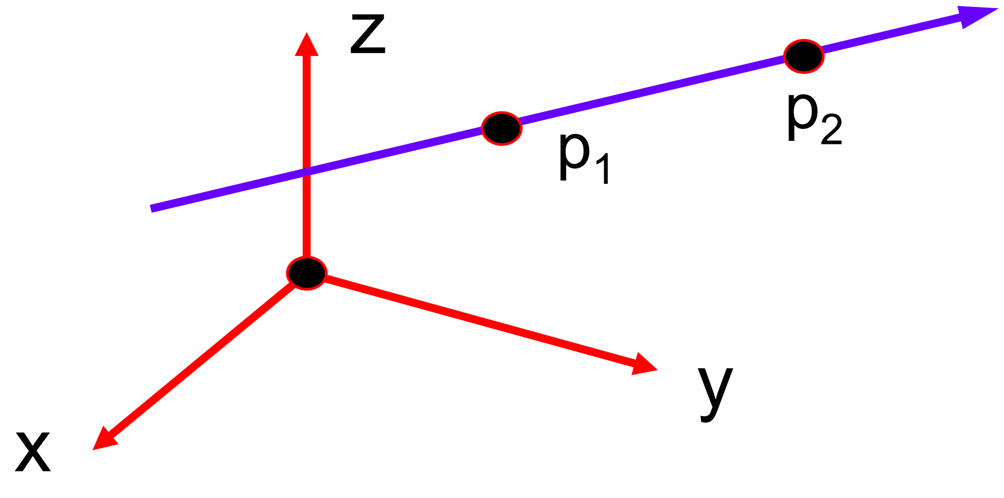
- Let's check the transformation of the X axis under M^{-1}:

$$M^{-1}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}u_x & v_x & w_x & 0\\u_y & v_y & w_y & 0\\u_z & v_z & w_z & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}u_x\\u_y\\u_z\\1\end{bmatrix} = U$$

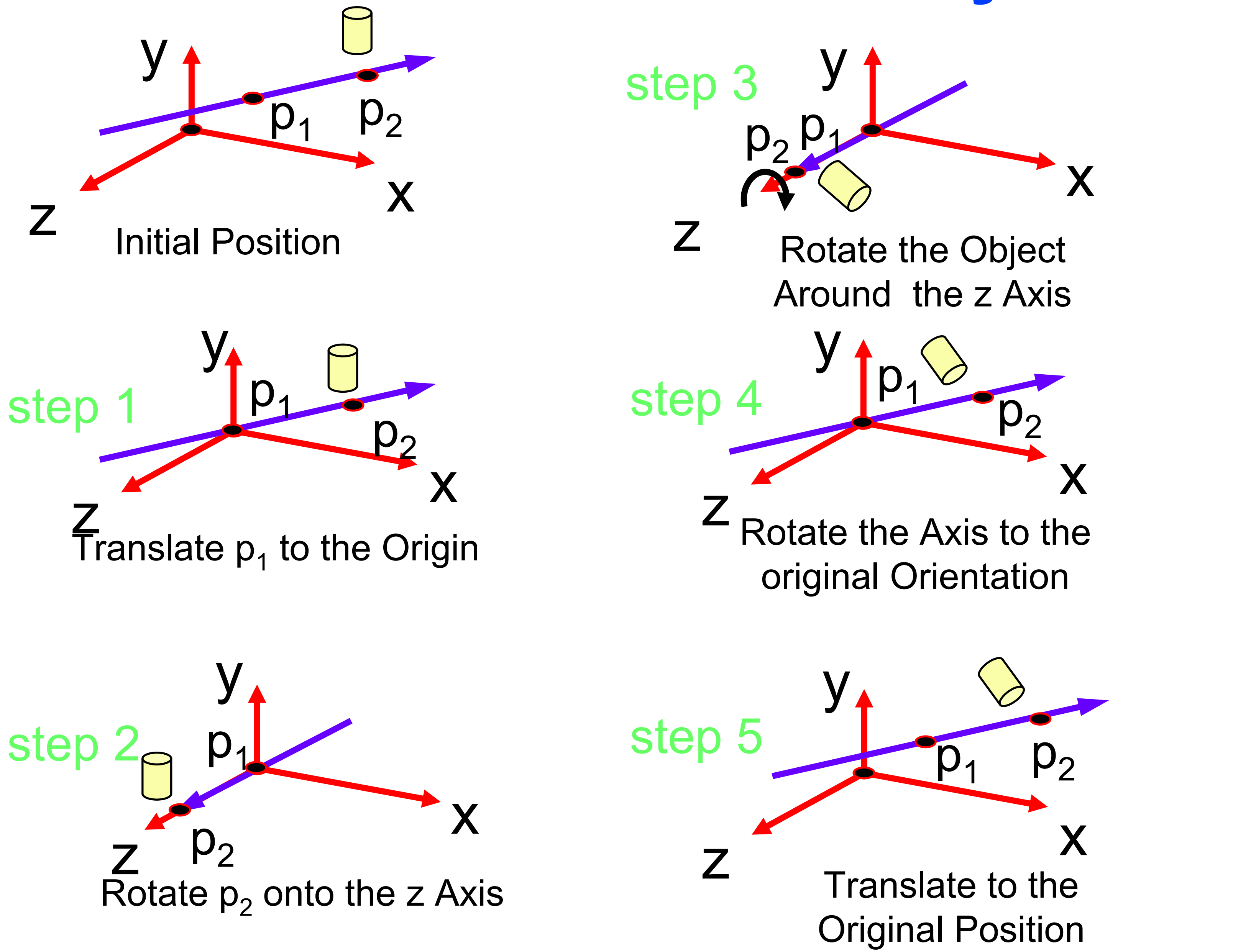
- Similarly, Y goes into V, and Z goes into W

Rotation About an Arbitrary Axis

- Axis of rotation can be located at any point: 6 d.o.f. (we must specify 2 points p_1 and p_2)
- **The idea**: make the axis coincident with one of the coordinate axes (z axis), rotate by θ, and then transform back



Rotation About an Arbitrary Axis



Rotation About an Arbitrary Axis

- Step1: $$T = \begin{pmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- Step 2: $$M = \begin{pmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- Step 3: $$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- Step 4: $$M^{-1} = \begin{pmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- Step 5: $$T^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- Composition: $P' = T^{-1} M^{-1} R M T P$

Rotation About Arbitrary Axis

- Constructing an orthonormal system along the rotation axis:
  - A vector *W* parallel to the rotation axis:

$$s = p_2 - p_1; \qquad w = \frac{s}{|s|}$$

  - A vector *V* perpendicular to *W*:

$$a = w \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad ; \quad v = \frac{a}{|a|}$$

  - A vector *U* forming a right-handed orthogonal system with *W* and *V*:

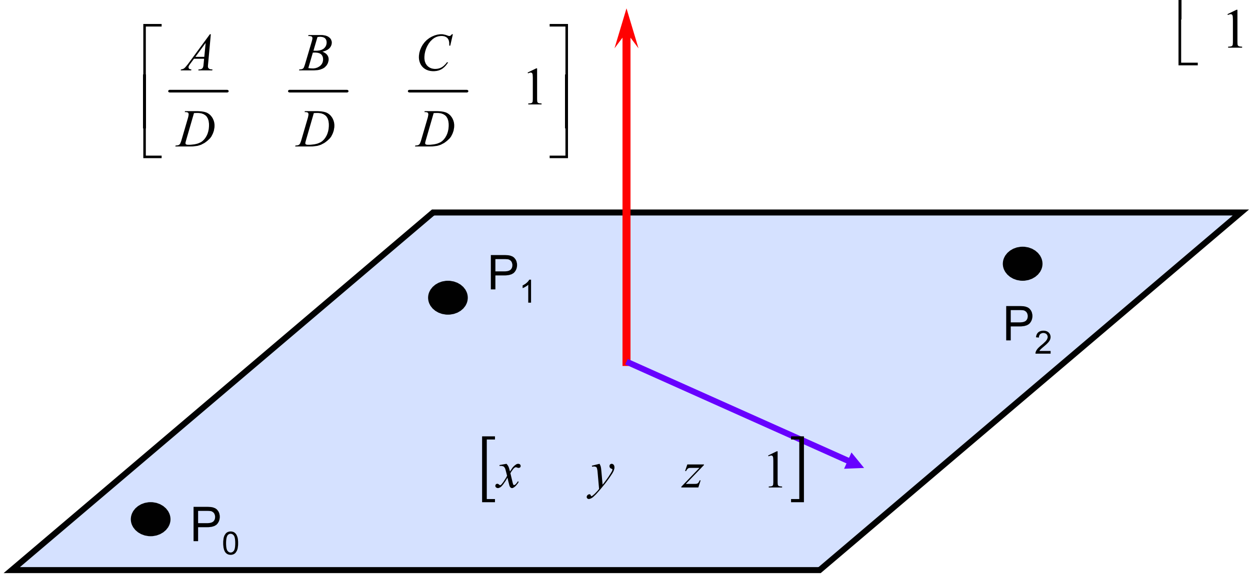
$$u = v \times w$$

Transforming Planes

- Plane representation:
  - By three non-collinear points
  - By implicit equation:

$$Ax + By + Cz + D = \begin{bmatrix} A & B & C & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{A}{D} & \frac{B}{D} & \frac{C}{D} & 1 \end{bmatrix}$$



Transforming Planes

- One way to transform a plane is by transforming any three non-collinear points on the plane

- Another way is to transform the plane equation: Given a transformation *T* such that

$$T\,[x, y, z, 1] = [x', y', z', 1]$$

find [A',B',C',D'], such that:

$$\begin{bmatrix} A' & B' & C' & D' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = 0$$

Transforming Planes

- Note that:

$$\begin{bmatrix} A & B & C & D \end{bmatrix} T^{-1} T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

- Thus, the transformation that we should apply to the plane equation is:

$$\begin{bmatrix} A' & B' & C' & D' \end{bmatrix} = \begin{bmatrix} A & B & C & D \end{bmatrix} T^{-1}$$