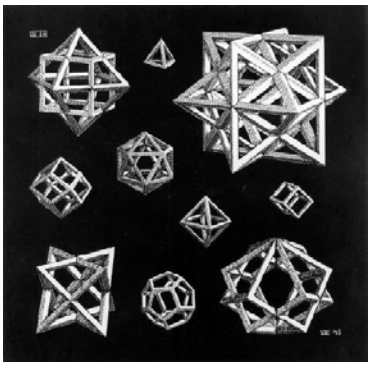
Solid Modeling

Foley & Van Dam, Chapter 11.1 and Chapter 12

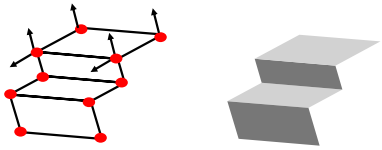


Solid Modeling

- Polygon Meshes
- Plane Equation and Normal Vectors
- Volume Representation
  - Sweep Volume
  - Spatial Occupancy Enumeration
  - Binary Space Partition Tree
  - Constructive Solid Geometry
  - Boundary Representation

Polygon Meshes

- A polygon mesh is a collection of polygons, along with a normal vector associated to each polygon vertex :
  - An edge connects two vertices
  - A polygon is a closed sequence of edges
  - An edge can be shared by two adjacent polygons
  - A vertex is shared by at least two edges
  - A normal vector pointing "outside" is associated with each polygon vertex



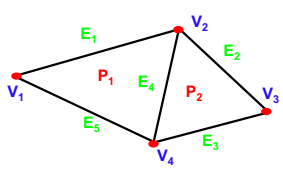
Polygon Meshes

Properties:

- Connectedness: A mesh is connected if there is an path of edges between any two vertices
- Simplicity: A mesh is simple if the mesh has no holes in it
- Planarity: A mesh is planar if every face of it is a planar polygon
- Convexity: The mesh is convex if the line connecting any two points in the mesh belongs to the mesh

Representing Polygon Meshes

- **Explicit (vertex list)**
  - $P_1=(V_1,V_2,V_4)$
  - $P_2=(V_2,V_3,V_4)$
- **Pointers to a vertex list**
  - $V=(V_1,V_2,V_3,V_4)$
  - $P_1=(1,2,4)$
  - $P_2=(4,2,3)$
- **Pointers to an edge list**
  - $V=(V_1,V_2,V_3,V_4)$
  - $E1=(1,2,P_1,\lambda)$ λ Represents null
  - $E2=(2,3,P_2,\lambda)$
  - $E3=(3,4,P_2,\lambda)$
  - $E4=(4,2,P_1,P_2)$
  - $E5=(4,1,P1,\lambda)$
  - $P1=(E_1,E_4,E_5)$
  - $P2=(E_2,E_3,E_4)$



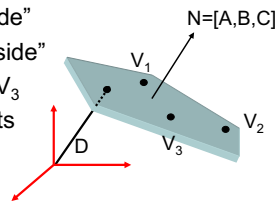
Representing Polygon Meshes

- Explicit
  - Space effective for single polygons
  - Not very informative
- Pointers to Vertex List
  - Saves spaces for shared vertices
  - Easy to modify a single vertex
  - Difficult to find polygons sharing an edge
  - Multiple drawing and clipping of shared edges
- Pointers to Edge List
  - Solves multiple drawing and clipping problem
  - Still hard to find all edges sharing a vertex

Plane Equations

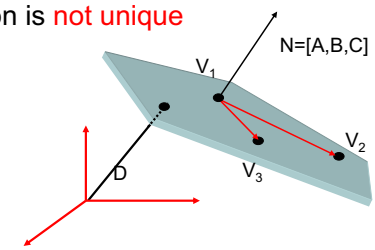
$$Ax + By + Cz + D = 0 \quad or \quad [A \quad B \quad C \quad D]\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

- N=[A,B,C] is the plane normal (|N|=1)
- D= -P∗N
- |D| is the plane distance from the origin.
- N points "outside", so:
  - •Ax+By+Cz+D<0 , (x,y,z) is "inside"
  - •Ax+By+Cz+D>0 , (x,y,z) is "outside"
- Three non collinear points V_1, V_2, V_3 are sufficient to find the coefficients A, B, C and D



Plane Equations

- Given V_1,V_2,V_3, the plane normal (or the coefficients A, B and C) can be computed with the cross product:
  $S=(V_3-V_1)x(V_2-V_1)$ and $N=S/|S|$
- D can be found by plugging any point of the plane in the equation Ax+By+Cz+D=0
- The plane equation is not unique



Plane Equations

- **Example:**

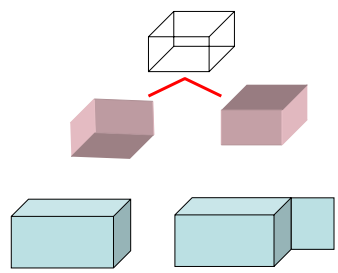
Find the equation of the plane containing the points
$V_1=(2,1,2)$, $V_2=(3,2,1)$ and $V_3=(1,1,3)$
First we compute the vectors
$V_2-V_1 = (1,1,-1)$ and $V_3-V_1= (-1,0,1)$
The A, B and C coefficients of the equation are given by the cross product $S=(V_2-V_1)x(V_3-V_1)$
$N = [A, B, C] = S/|S| = (1,0,1)/|(1,0,1)| = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$
Finally we substitute V_1 in the equation to find D.
To simplify the computation, we use [A,B,C]=(1,0,1)

$$Ax+By+Cz+D=0$$
$$1*2+0*1+1*2+D=0$$
$$D=-4$$

And the plane equation is x+z-4=0
The equation with the unitary normal is $\frac{1}{\sqrt{2}}(x+z-4)=0$

Volume Representation

- A collection of techniques to represent and define volumetric objects
- Desired properties:
  - Rich representation
  - Unambiguous
  - Unique representation
  - Accurate
  - Compact
  - Efficient
  - Possible to test validity



Volume Representation

- Volume Representation
  - Primitive Instancing
  - Sweep volumes
  - Spatial Occupancy (voxels, Octree, BSP)
  - Constructive Solid Geometry
- Boundary Rep.
  - Polyhedra
  - Free form

Primitive Instancing

- Define a family of parameterized objects
- The definition is procedural (a routine defines it)
- Not general, must be individually defined for each family of objects

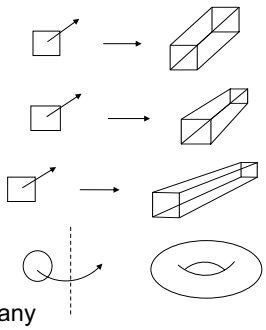
Example: Wheels/Gears

Diameter = 10
Teeth = 16
Hole = 3

Diameter = 8
Teeth = 24
Hole = 5

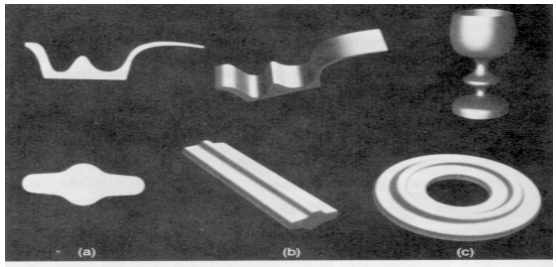
Sweep Volume

- Sweep Volume: sweeping a 2D area along a trajectory creates a new 3D object
- Translational Sweep: 2D area swept along a linear trajectory normal to the 2D plane
- Tapered Sweep: scale area while sweeping
- Slanted Sweep: trajectory is not normal to the 2D plane
- Rotational Sweep: 2D area is rotated about an axis
- General Sweep: object swept along any trajectory and transformed along the sweep



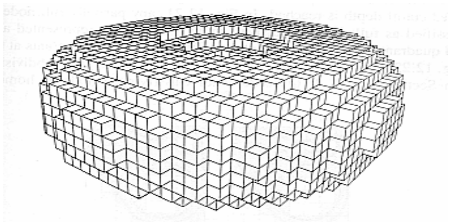
Sweep Volume

Translational and rotational sweep volumes



Spatial Occupancy Enumeration

- Space is described as a regular array of cells (usually cubes). Each cell is called a ***Voxel***
- A 3D object is represented as a list of filled voxels

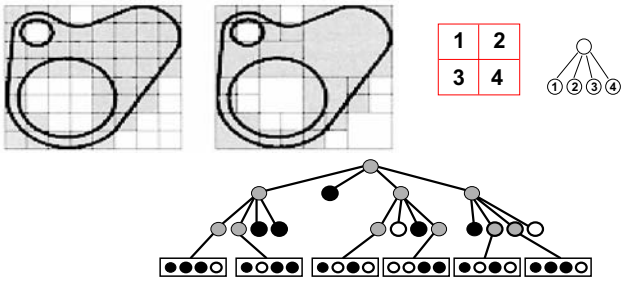


Spatial Occupancy Enumeration

- **Pros:**
  - Easy to verify if a point (a voxel) is inside or outside an object
  - Boolean operations are easy to apply
- **Cons:**
  - Memory costs are high
  - Resolution is limited to size and shape of voxel

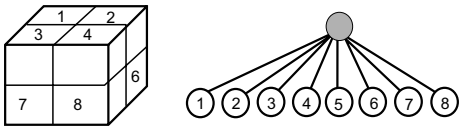
Quadtrees

- A ***Quadtree*** is a data structure enabling efficient storage of 2D data
- Completely full or empty regions are represented by one cell; recursive subdivision is used on the others

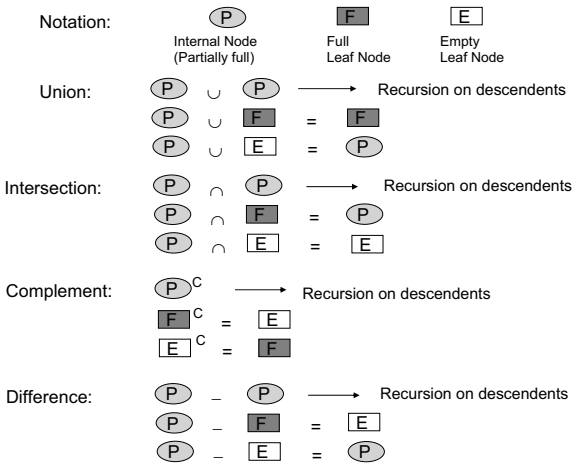


Octrees

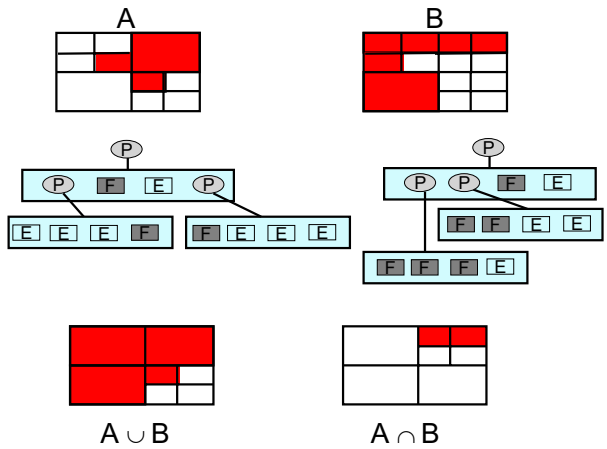
- An **Octree** is a 3D generalization of a Quadtree
- Each node in an Octree has eight children rather than four
- Describes a recursive partitioning of a volume into cells that are completely full or empty



Binary Operations on Quad/Octrees

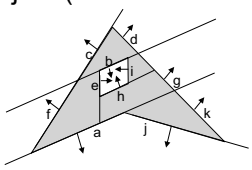


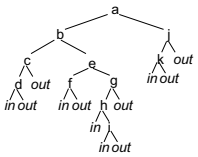
Binary Operations on Quad/Octrees



Binary Space Partition Trees - BSP

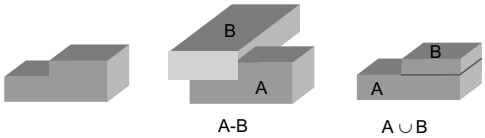
- Each internal node represents a plane in 3D space
- Each node has 2 children pointers one for each side of the plane.
- A leaf node represents a homogeneous portion of space - either “in” or “out”.
- Easy to determine if a point is inside or outside an object (recurse down the BSP tree)





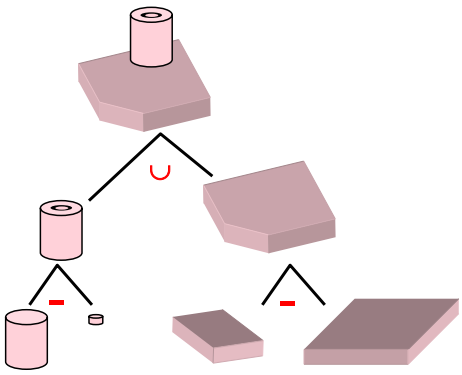
Constructive Solid Geometry

- Combine simple primitives using Boolean operations and represent as a binary tree.
- To generate the object the tree is processed in a depth-first pass.
- Cons: representation is not unique



Constructive Solid Geometry

An object defined by a binary CSG tree

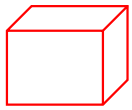


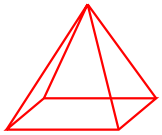
Boundary Representations

- A closed 2D surface defines a 3D object
- At each point on the boundary there is an “in” and an “out” side
- Boundary representations can be defined in two ways:
  - Primitive based. A collection of primitives forming the boundary (polygons, for example)
  - Freeform based (splines, parametric surfaces, implicit forms)

Boundary Representations

- A polyhedron is a solid bounded by a set of polygons
- A polyhedron is constructed from:
  - Vertices V
  - Edges E
  - Faces F
- Each edge must connect two vertices and be shared by exactly two faces
- At least three edges must meet at each vertex

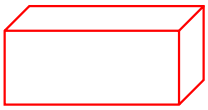




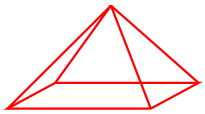
Boundary Representations

- A **simple polyhedron** is one that can be deformed into a sphere (contains no holes)
- A simple polyhedron must satisfies *Euler's formula:*

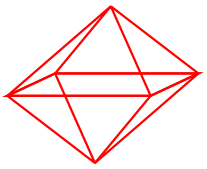
$$V-E+F=2$$



V = 8
E = 12
F = 6



V = 5
E = 8
F = 5



V = 6
E = 12
F = 8

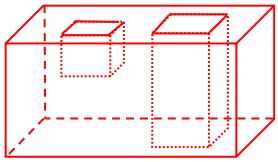
Boundary Representations

- Euler's formula can be generalized to a polyhedron with holes and multiple components

$$V-E+F-H=2(C-G)$$

Where:

- H is the number of holes in the faces
- C is the number of separate components
- G is the number of pass-through holes (genus if C=1)
- V, E and F are respectively vertices, edges and faces



$$V - E + F - H = 2\,(C - G)$$

$$24 - 36 + 15 - 3 = 2\,(1 - 1)$$