Rhombic Polyhedra Puzzles Part 1

by George Bell and
Stephen Chin

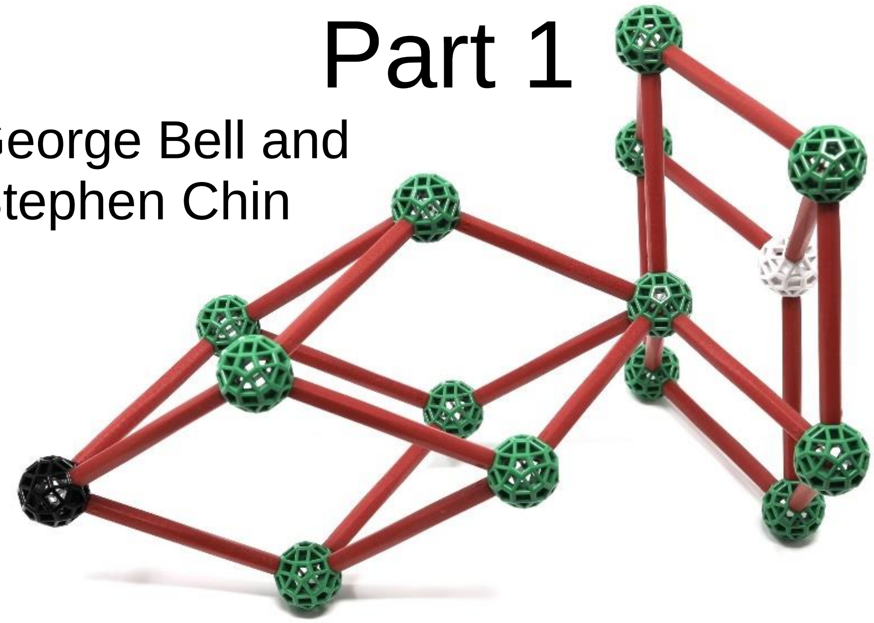

Figure 1. Fat and thin rhombs connected using Zometool [1]. The black (white) vertex will become the triacontahedron south pole (north pole).

Introduction

Golden rhombohedra (Figure 1) make fascinating building blocks for puzzles**,** we'll refer to them as **rhombs**. They come in two varieties: **Prolate** and **Oblate**. These terms can be hard to remember so we'll use **Fat** (**F**) and **Thin** (**T**). Imagine a wireframe cube with flexible joints. To make each rhomb, we squash the cube so that every face becomes a golden rhombus (the ratio of the diagonals is the golden ratio $\phi = (\sqrt{5}+1)/2 \cong 1.618$). Rhombs have many wonderful properties and have been called the 3D analogue of Penrose tiles [2].

One reason puzzles made from rhombs are interesting is because they are so far removed from the familiar cube-based geometry. A nice way to explore this geometry is using Zometool [1], any rhombic polyhedron can be constructed using only red struts. You can even build these puzzles using Zometool, but unfortunately it is not possible to assemble them because the pieces have duplicated vertices and edges. Making workable puzzles has been the domain of the master woodworker, consequently wood copies are rare and highly valued.

In Part 1 we'll discuss the geometry of these puzzles, and introduce the basic puzzles created by gluing together rhombs. In the 1980's, Wayne Daniel created more complex puzzles by cutting each rhomb into identical or mirror image halves. In Part 2, we'll present a detailed accounting of split-rhomb puzzles.

Puzzle Geometry

Figure 2 shows four polyhedra which can be constructed out of rhombs, summarized in Table 1. These polyhedra are normally preceded by the words “golden” and “rhombic”. In this article we’ll often leave these qualifiers out. Note that the Figure 2 dodecahedron is NOT the common rhombic dodecahedron where the ratio of the diagonals is $\sqrt{2}$. The Figure 2 dodecahedron is often called a “rhombic dodecahedron of the second kind” or “Bilinski dodecahedron” [3].

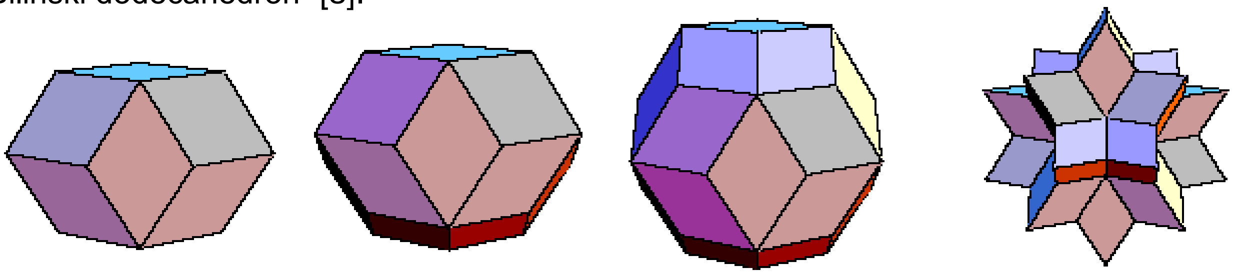

Figure 2. Rhombic polyhedra: dodecahedron, icosahedron, triacontahedron and hexecontahedron.

rhombic polyhedron	external			# of intern. vertices	# of rhomb decomps	# of internal rhombs
	faces	verts	edges			
dodecahedron	12	14	24	1	1 (2F + 2T)	0
icosahedron	20	22	40	4	1 (5F + 5T)	1T
triacontahedron	30	32	60	10	2 (10F + 10T)	1F + 3T
hexecontahedron	60	62	120	1	1 (20 F)	0

Table 1. Properties of rhombic polyhedra.

An **internal vertex** is not part of the basic polyhedron, but is a vertex in the decomposition into rhombs. Likewise, an **internal rhomb** is one which contributes no face of the final polyhedron.

The (Bilinski) dodecahedron can be decomposed into 2 fat (red, blue) and 2 thin (green, yellow) rhombs, as shown in Figure 3. Note that this decomposition has a directionality to it, with the fat rhombs on the left and the thin rhombs on the right, as indicated by the arrow.

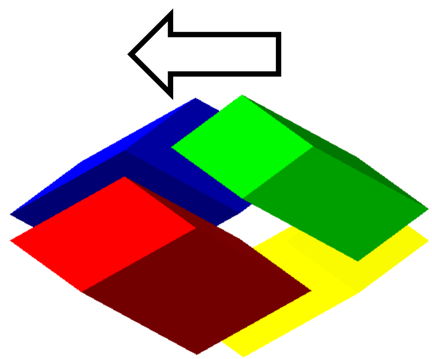

Figure 3. A (Bilinski) dodecahedron decomposed into 2F + 2T.

There are two known decompositions of the triacontahedron into rhombs (plus their mirror images, not counted as different). Any decomposition must have exactly one internal F. The long F diagonal is 80% of the triacontahedron diameter, making it difficult to fit even one inside! The internal F “core” defines the axis between N and S poles, perched on top is a thin rhomb defining the N pole (see Figures 1 and 4). Figure 4 (right) show the further decomposition of the triacontahedron into the N pole, three dodecahedra plus S pole “leftovers” (including the core).

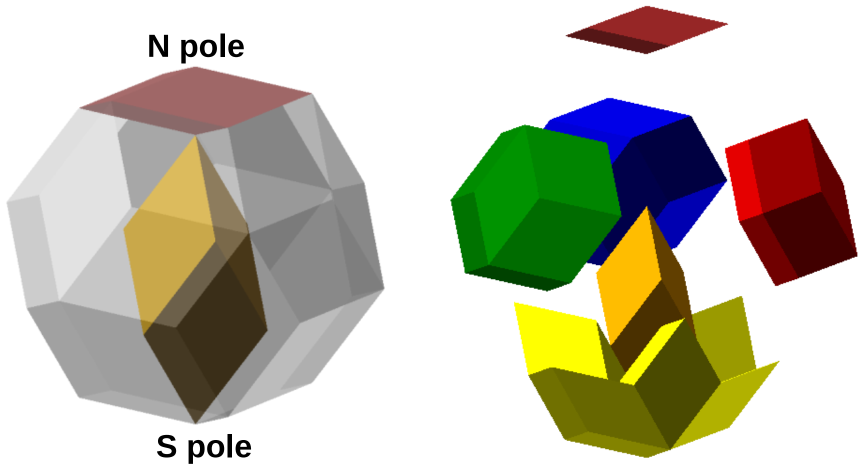

Figure 4. Triacontahedron core plus N pole rhomb; a triacontahedron decomposed into three dodecahedra plus N and S pole pieces.

Remember that the dodecahedron decomposition had a directionality to it. If we further decompose the three dodecahedra in Figure 4, there are two possibilities, diagrammed in Figure 5:

1. The directionality (arrows) go around the same direction viewed from the N pole. This leads to triacontahedron decomposition with 3-fold symmetry.
2. One arrow points opposite the other two. This leads to a triacontahedron decomposition with no overall symmetry. But, rather remarkably, we can add six (light blue) rhombs to the green dodecahedron to form the icosahedron! What remains is a yellow end cap (10 rhombs). The yellow end cap decomposition has 3-fold symmetry, but about a different axis.

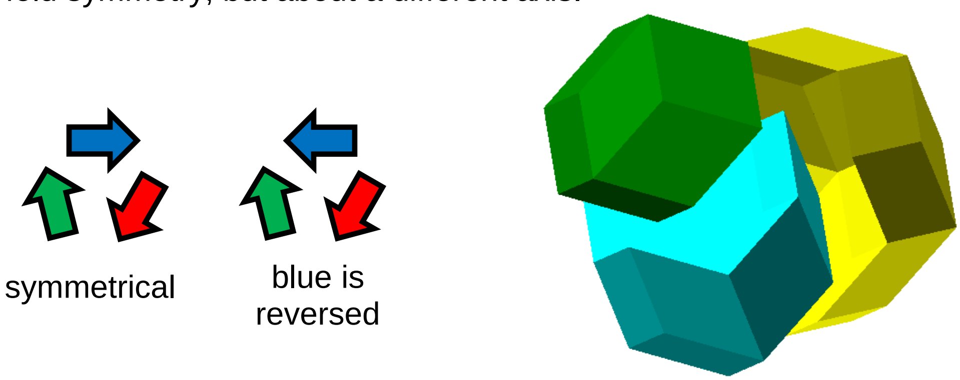

Figure 5. Two options for the directionality of the Figure 4 dodecahedra. The nonsymmetrical configuration leads to the icosahedron (green + light blue).

Curiously, all measurements of these polyhedra involve the Golden Ratio. The fat rhomb can be defined by the three edge vectors: $(\phi, \phi^2, 0)$, $(-\phi, \phi^2, 0)$ and $(0, \phi, \phi^2)$. The thin rhomb uses the same first two edge vectors plus $(-\phi^2, 0, \phi)$. Each face has a long diagonal of length $2\phi^2$, a short diagonal of length 2ϕ, and an area of $2\phi^3$.

The rhomb edge length is $e = \phi\sqrt{\phi + 2} \cong 3.078$. The fat rhomb has volume $2\phi^5$, and the thin rhomb $2\phi^4$. Using Table 1 and the identity $\phi^2 = \phi + 1$ we can determine the volume of the dodecahedron ($4\phi^6$), icosahedron ($10\phi^6$), triacontahedron ($20\phi^6$) and hexecontahedron ($40\phi^5$).

The lengths above are in unscaled units. To make puzzles, we will scale all lengths by a scale factor *s*. To choose *s* one needs to know the desired puzzle size. The face to opposite face distance is $4\phi + 2 \cong 8.47$ for the triacontahedron, and $3\phi + 2 \cong 6.85$ for the icosahedron. If you are 3D printing you can apply your own scale factor to print the puzzle any size you like. On wood puzzles, the face to face distance can be measured with calipers to determine the scale factor (and edge length). You can measure the edge length directly, but it is difficult to get an accurate value.

Exact values for the vertices for these polyhedra can be found in the supplementary material for this CFF article. After the vertices are entered into a program, the edges are easily identified because rhombic polyhedra have equal edge lengths. Including the internal vertices gives the decomposition into rhombs. One can move between the two triacontahedron decompositions by moving a single internal vertex.

A few of the puzzles listed below have names—many do not. To classify the designs, we use the designers initials, followed by "RI" for a rhombic icosahedron puzzle or "RT" for a rhombic triacontahedron, followed by a number. For designers we use WD (Wayne Daniel), AG (Albert Gübeli), RB (Rik Brouwer), SC (Stephen Chin) and GB (George Bell).

Icosahedron Puzzles

The icosahedron has one decomposition into 10 rhombs, this decomposition contains a (Bilinski) dodecahedron but has no overall symmetry. There is always one thin rhomb which is internal. For the purposes of orienting these puzzles, it is often useful to identify this internal thin rhomb.

By gluing together 10 rhombs, it is not easy to make an interlocking puzzle. We have found four 3-piece designs: Albert Gübeli designed *Aurels* (AGRI01) in 2004, and Stephen Chin SCRI01 – SCRI03 in 2016 (Figure 6). All have the property that any piece can come out first. SCRI03 was Stephen Chin's exchange puzzle at IPP36 (Kyoto, 2016), with the name: *Golden Rhombic Icosahedron*.

Figure 6. SCRI02 (assembled), SCRI03 (pieces), s=9.2 mm. Made by S. Chin.

Triacontahedron Puzzles

A good triacontahedron puzzle to begin with is *Mateos* (AGRT01), designed by Albert Gübeli in 1989 (Figure 7). Start with three 4-rhomb batman pieces (so named because of the similarity to the infamous superhero logo). Each batman piece takes 1F + 1T from the bottom yellow S pole in Figure 4 plus 1F + 1T from each dodecahedron. What remains (the yellow piece in Figure 7) are 8 rhombs combining N pole and core with 3F and 3 internal T, this piece has 3-fold symmetry. Albert Gübeli found (rather remarkably) that this yellow piece can be cut into two mirror image halves (4 rhombs each).

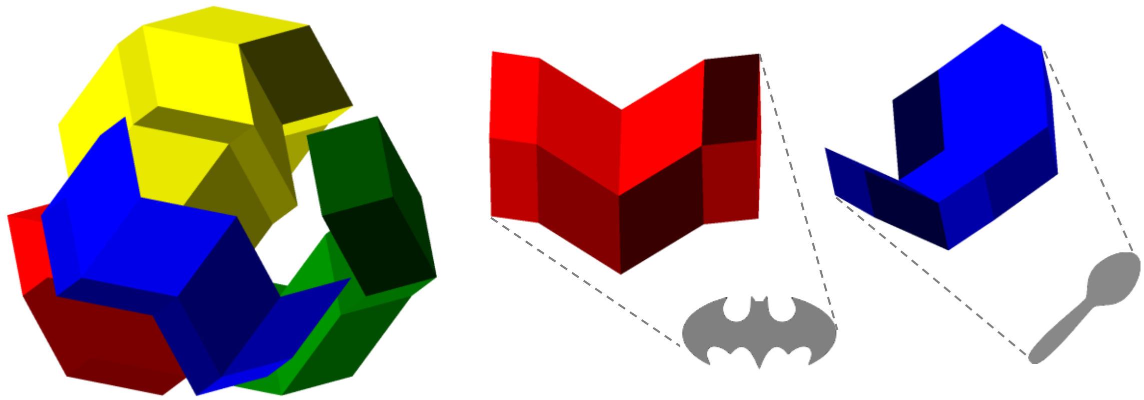

Figure 7. *Mateos* (AGRT01), "batman" piece, "spoon" piece.

Mateos has five pieces and can be assembled in two mirror image ways. The batman pieces are added to the two core pieces. Interestingly, the assembly does not interlock until the final piece is inserted, even though the last 3 pieces are identical and can be inserted in any order. This is a hint that coordinate motion designs lurk nearby.

Mateos can be modified in several ways to create three and four piece coordinate motion puzzles. First, we can transfer two more rhombs to the batman piece to form a 6-rhomb "spoon" piece (Figure 7). Three spoons go together with coordinate motion, but the assembled puzzle is missing the N pole and core.

There are several ways to modify the spoon piece to form the full triacontahedron. One solution is to add the N pole to one spoon, the core rhomb to the second, and leave the third alone. This results in a coordinate motion puzzle GBRT01 where all three pieces are different. Another possibility is to cut both missing rhombs into three identical pieces (there are several ways to do this). Adding one third to each spoon results in a coordinate motion puzzle of three identical pieces, GBRT02.

ID	Designer	Year	Pcs	Co-Mo?	Comments
AGRT01	Albert Gübeli	1989	5	no	*Mateos*
RBRT01	Rik Brouwer	2006	4	yes	*Triakon* [2]
AGRT02	Albert Gübeli	2015	4	yes	*Soccerit,* IPP35
SCRT01	Stephen Chin	2016	4	no	Uses non-symmetrical decomp.
SCRT02	Stephen Chin	2016	4	no	
GBRT01	George Bell	2016	3	yes	
GBRT02	George Bell	2016	3	yes	Three identical pieces.
GBRT03	George Bell	2016	4	yes	

Table 2. Triacontahedron puzzles using full rhombs (sorted by year).

A different strategy is to move one rhomb from the yellow end cap to each batman piece, we end up with 4 pieces of 5 rhombs each. There are many ways to do this, but one option results in Rik Brouwer's coordinate motion puzzle *Triakon* [2], classified as RBRT01. Another option (GBRT03) yields three pieces plus what remains of the yellow cap (4 rhombs) which retains 3-fold symmetry. The three pieces go together using coordinate motion and the final piece locks them in place.

Figure 8. SCRT01 (assembled), SCRT02 (pieces), s=9.2 mm. Made by S. Chin.

There are also modifications of *Mateos* which are serially interlocking. An example is SCRT02, where the batman piece is the last to be go in (Figure 8). *Soccerit* (AGRT02) is a totally new 4-piece design by Albert Gübeli where one of the pieces contains a hole. All the designs above use the symmetrical decomposition of the triacontahedron into rhombs. SCRT01 is the exception, it is a 4-piece design based on the non-symmetrical decomposition. All these puzzles are summarized in Table 2.

Summary

Rhombs are useful building blocks for puzzles, the target shapes are primarily the rhombic icosahedron and rhombic triacontahedron (hexecontahedron, anyone?). These puzzles are challenging to make in wood, and are consequently rare and expensive. Please contact George Bell (gibell@comcast.net) if you would like stl files for 3D printing personal copies of these puzzles. Stephen Chin may also have a few spare wood copies available (stephenchin57@gmail.com).

References

[1] Zometool, http://www.zometool.com/
[2] Rik Brouwer, More Icosahedral Fun, CFF 69, March 2006, pp 18-22.
[3] Bilinski, S. "Über die Rhombenisoeder", *Glasnik Mat. Fiz. Astr.,* **15**, 251-63, 1960.